QUIZZES AND EXAMS FOR MATH 1310
ENGINEERING CALCULUS I

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Week 1 Quiz

Problem. Let

$$\begin{aligned} f(x) &= \sqrt{4-2^x}, \\ h(x) &= \sqrt{4-x}. \end{aligned}$$

a) (*2 points.*) Find the domain of $f(x)$.

b) (*2 points.*) Find the domain of $h(x)$.

c) (*4 points.*) Find the inverse function of $f(x)$.

d) (*2 points.*) Find a function g such that $f \circ g = h$.

Week 2 Quiz

Problem 1. *(6 points.)* Sketch the graph of the following function

$$f(x) := \begin{cases} e^x & \text{if } 0 \leq x < 1 \\ \lfloor x \rfloor & \text{if } 1 \leq x \leq 3 \\ \cos\left(\frac{\pi}{3}x\right) & \text{if } x > 3 \end{cases}.$$

Then find each of the following, or state that does not exist:

$$\begin{array}{llll} \lim_{x\to 1^-} f(x) = \dots & \lim_{x\to 1^+} f(x) = \dots & \lim_{x\to 1} f(x) = \dots & f(1) = \dots \\ \lim_{x\to 2^-} f(x) = \dots & \lim_{x\to 2^+} f(x) = \dots & \lim_{x\to 2} f(x) = \dots & f(2) = \dots \\ \lim_{x\to 3^-} f(x) = \dots & \lim_{x\to 3^+} f(x) = \dots & \lim_{x\to 3} f(x) = \dots & f(3) = \dots. \end{array}$$

Date: Fall 2015.

Problem 2. *(4 points.)* Evaluate the following limit

$$\lim_{t\to 0} \frac{t^2}{2-\sqrt{4+t^2}}.$$

Week 3 Quiz

Problem 1. i) *(4 points.)* Compute the following limits

$$\lim_{x\to(\frac{\pi}{2})^+} \frac{x}{\cos(x)}, \qquad \lim_{x\to(\frac{\pi}{2})^-} \frac{x}{\cos(x)}.$$

ii) *(2 points.)* Find the vertical asymptotes to the graph of the function $y = \frac{x}{\cos(x)}$.

Problem 2. *(4 points.)* Compute the following limit using the squeeze theorem

$$\lim_{x\to 0} (e^x - 1)\cos\left(\frac{1}{x^2}\right).$$

Super Quiz 1

Problem 1. *(3 points.)* Find the horizontal and vertical asymptotes of

$$y = \frac{1-x^2+3x}{4x^2-8x-12}.$$

Problem 2. *(5 points.)* Find the following limit

$$\lim_{t\to\infty} \left(t - \sqrt{t^2+4t}\right).$$

Problem 3. i) *(5 points.)* Find the derivative of

$$f(x) = \sqrt{3x-1}.$$

ii) *(2 points.)* Find the tangent line to $y = \sqrt{3x-1}$ at $(\frac{5}{3}, 2)$.

Problem 4. *(5 points.)* Find the following limit

$$\lim_{x\to 0^-} e^{\frac{1}{x}} \sin\left(\frac{1}{x}\right).$$

Week 5 Quiz

Problem 1. *(6 points.)* Find the tangent line to $y = \frac{\sqrt{x}}{e^x}$ at $(4, \frac{2}{e^4})$.

Problem 2. *(4 points.)* Compute the following limit

$$\lim_{x\to 0} \frac{1 - 2015x - (1-x)^{2015}}{x^2}.$$

Midterm 1

Problem 1. Sketch the graph of the following function

$$f(x) := \begin{cases} 2^{-x} & \text{if } 0 \le x \le 2 \\ \lfloor x \rfloor & \text{if } 2 < x \le 4 \\ \frac{1}{4-x} & \text{if } x > 4 \end{cases}.$$

i) *(2 points.)* Find the vertical and horizontal asymptotes to the graph of f.

Solution. Since

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{4-x} = 0,$$

the x-axis is a horizontal asymptote. The function $f(x)$ has only jump discontinuities on the interval $[0, 4)$ and is continuous on $(4, \infty)$, hence there is at most one vertical asymptote, namely $x = 4$. Since

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} = \frac{1}{4-x} = -\infty,$$

the line $x = 4$ is indeed a vertical asymptote. □

ii) *(12 points.)* Find each of the following, or state that it does not exist:

$$\lim_{x\to 2^-} f(x) = \ldots \qquad \lim_{x\to 2^+} f(x) = \ldots \qquad \lim_{x\to 2} f(x) = \ldots \qquad f(2) = \ldots$$

$$\lim_{x\to 3^-} f(x) = \ldots \qquad \lim_{x\to 3^+} f(x) = \ldots \qquad \lim_{x\to 3} f(x) = \ldots \qquad f(3) = \ldots$$

$$\lim_{x\to 4^-} f(x) = \ldots \qquad \lim_{x\to 4^+} f(x) = \ldots \qquad \lim_{x\to 4} f(x) = \ldots \qquad f(4) = \ldots.$$

Solution.

$$\begin{array}{llll}
\lim_{x\to 2^-} f(x) = \frac{1}{4} & \lim_{x\to 2^+} f(x) = 2 & \lim_{x\to 2} f(x) = DNE & f(2) = \frac{1}{4} \\
\lim_{x\to 3^-} f(x) = 2 & \lim_{x\to 3^+} f(x) = 3 & \lim_{x\to 3} f(x) = DNE & f(3) = 3 \\
\lim_{x\to 4^-} f(x) = 3 & \lim_{x\to 4^+} f(x) = -\infty & \lim_{x\to 4} f(x) = DNE & f(4) = 4.
\end{array}$$

□

Problem 2. i) *(4 points.)* Find the horizontal and vertical asymptotes to the graph of

$$f(x) = \frac{e}{\sin(2x)},$$

if any.

Solution. Note that

$$\lim_{x\to\infty} \frac{e}{\sin(2x)} = \frac{e}{\lim_{x\to\infty}\sin(2x)}.$$

The limit in the denominator on the right-hand side does not exist, hence the limit on the left-hand side does not exist. The same holds for the limit for $x \to -\infty$. Hence there are no horizontal asymptotes. We have a vertical asymptote when

$$\sin(2x) = 0 \quad \Leftrightarrow \quad 2x = k\cdot\pi \quad \text{for } k \in \mathbb{Z} \quad \Leftrightarrow \quad x = \frac{k\cdot\pi}{2} \quad \text{for } k\in\mathbb{Z}.$$

□

ii) *(5 points.)* Find the domain of the following function

$$f(x) = \frac{\sqrt{x^2-1}}{x\cdot\ln(x)}.$$

Solution. The domain of $\sqrt{x^2-1}$ is $(-\infty,-1]\cup[1,\infty)$. The domain of $\ln(x)$ is $(0,\infty)$. Moreover, we require the denominator to be non-zero, that is, $x \neq 0$ and $x \neq 1$. The intersection of all such sets is $(1,\infty)$. □

Problem 3. i) *(4 points.)* Compute the following limit

$$\lim_{x\to\infty} \sqrt[3]{\frac{(9x-3)(3-6x)}{(4+x)(2x+1)}}.$$

Solution. We have

$$\begin{aligned}
\lim_{x\to\infty} \sqrt[3]{\frac{(9x-3)(3-6x)}{(4+x)(2x+1)}} &= \sqrt[3]{\lim_{x\to\infty} \frac{(9x-3)(3-6x)}{(4+x)(2x+1)}} \\
&= \sqrt[3]{\lim_{x\to\infty} \frac{-54x^2}{2x^2}} \\
&= \sqrt[3]{-27} = -3.
\end{aligned}$$

□

ii) *(4 points.)* Compute the following limit

$$\lim_{x\to\infty} (\sqrt{4x^2+3x} - 2x).$$

Solution. We have

$$\begin{aligned}
\lim_{x\to\infty} (\sqrt{4x^2+3x} - 2x) &= \lim_{x\to\infty} \left((\sqrt{4x^2+3x} - 2x) \cdot \frac{\sqrt{4x^2+3x}+2x}{\sqrt{4x^2+3x}+2x} \right) \\
&= \lim_{x\to\infty} \frac{4x^2+3x-4x^2}{\sqrt{4x^2+3x}+2x} \\
&= \lim_{x\to\infty} \frac{3x}{\sqrt{4x^2+3x}+2x} \\
&= \lim_{x\to\infty} \frac{3}{\sqrt{4+3\frac{1}{x}}+2} = \frac{3}{\sqrt{4}+2} = \frac{3}{4}.
\end{aligned}$$

□

Problem 4. i) *(5 points.)* Compute the following limit

$$\lim_{x\to 0} \left(\frac{1-\cos(3x)}{x} \cdot \sin\left(\frac{3}{x^2}\right) \right).$$

Solution. Note that $\sin\left(\frac{3}{x^2}\right)$ is a bounded function, while

$$\lim_{x\to 0} \frac{1-\cos(3x)}{x} = 3 \lim_{x\to 0} \frac{1-\cos(3x)}{3x} = 3 \cdot 0 = 0.$$

It follows that the product of the two functions has limit 0 (since “bounded $\cdot$ 0 = 0”). Alternatively, one can use the squeeze theorem. □

ii) *(5 points.)* Compute the following limit

$$\lim_{x\to 0} \quad \frac{\sin(3x)\cdot\cos^2(2x)}{\sin(2x)\cdot e^{\sin(x^2)}}.$$

Solution. We have

$$\begin{aligned}
\lim_{x\to 0} \quad \frac{\sin(3x)\cdot\cos^2(2x)}{\sin(2x)\cdot e^{\sin(x^2)}} &= \left(\lim_{x\to 0} \quad \frac{\sin(3x)}{\sin(2x)}\right)\left(\lim_{x\to 0} \quad \frac{\cos^2(2x)}{e^{\sin(x^2)}}\right) \\
&= \left(\lim_{x\to 0} \quad \frac{\sin(3x)}{\sin(2x)}\right)\cdot 1 \\
&= \left(\lim_{x\to 0} \quad 3\cdot\frac{\sin(3x)}{3x}\cdot\frac{1}{2}\frac{2x}{\sin(2x)}\right) = \frac{3}{2}.
\end{aligned}$$

□

Problem 5. *(9 points.)* Find the equations of the two lines passing through $(0, 13)$ and tangent to $y = 9 - x^2$.

Solution. On one hand, the slope of the tangent line to $y = 9 - x^2$ at the point $(x, y(x))$ is $y' = -2x$. On the other hand, the slope of the line passing through the points $(x, y(x))$ and $(0, 13)$ is $\frac{y(x)-13}{x-0}$, where $y(x) = 9 - x^2$. Equating the two slopes, we have

$$-2x = \frac{(9 - x^2) - 13}{x - 0}.$$

Multiplying by x, we have

$$-2x^2 = -x^2 - 4.$$

We deduce $x^2 = 4$, hence $x = 2$, or $x = -2$. It follows that one of the two desired tangent lines passes through the point $(2, y(2)) = (2, 5)$, and the other through the point $(-2, y(-2)) = (-2, 5)$. The tangent line passing through the point $(2, 5)$ is $y = -4x + 13$, and the tangent line passing through the point $(-2, 5)$ is $y = 4x + 13$. □

Problem 6. i) *(5 points.)* Find the points where the curve

$$y = \frac{x^3}{3} + x^2 - 3x$$

has an horizontal tangent line.

Solution. First we compute

$$y' = x^2 + 2x - 3.$$

Then we solve

$$x^2 + 2x - 3 = 0.$$

The two solutions are $x = -3$ or $x = 1$. □

ii) *(5 points.)* Find the derivative of

$$f(x) = e^{\cos(\sqrt[3]{3x})}.$$

Solution. We have

$$\begin{aligned} f' &= (e^{\cos(\sqrt[3]{3x})})' = e^{\cos(\sqrt[3]{3x})} \cdot (\cos(\sqrt[3]{3x}))' \\ &= e^{\cos(\sqrt[3]{3x})} \cdot (-\sin(\sqrt[3]{3x})) \cdot (\sqrt[3]{3x})' \\ &= e^{\cos(\sqrt[3]{3x})} \cdot (-\sin(\sqrt[3]{3x})) \cdot \left(\frac{1}{3}(3x)^{-\frac{2}{3}}\right) \cdot (3x)' \\ &= e^{\cos(\sqrt[3]{3x})} \cdot (-\sin(\sqrt[3]{3x})) \cdot \left(\frac{1}{3}(3x)^{-\frac{2}{3}}\right) \cdot 3. \end{aligned}$$

□

Week 9 Quiz

Problem 1. *(5 points.)* Find the linearization of the function $f(x) = \cos(x)$ at $a = \frac{\pi}{2}$, and use it to give an approximate value for $\cos(1.5)$ and $\cos(1.6)$. Say in each case if the approximation is an overestimate, or an underestimate. (Note: $\frac{\pi}{2} \approx 1.57$.)

Problem 2. *(5 points.)* Find the critical numbers of the following function

$$f(x) = x^{\frac{1}{3}}(4 - x).$$

Find the global maximum and minimum values of $f(x)$ on the interval $[-1, 8]$.

Super Quiz 2

Problem 1. *(9 points.)* i) Find the critical points of the following function

$$f(x) = 18x^2 + 8x^3 - 3x^4.$$

ii) Use the first derivative test to decide which critical points give a local maximum value and which give a local minimum value.
iii) Use the second derivative test to solve the previous problem.

Problem 2. *(5 points.)* Find the linearization of the function $f(x) = \sin(x)$ at $a = \pi$, and use it to give an approximate value for $\sin(3.1)$ and $\sin(3.2)$. Say in each case if the approximation is an overestimate, or an underestimate.

Problem 3. *(6 points.)* Find the following limit

$$\lim_{x\to 0} \frac{3-3\cos(2x)-6x^2+2x^4}{x^4}.$$

Week 11 Quiz

Problem 1. *(5 points.)* Find the following limit

$$\lim_{x\to 0^+} (\sin(x))^x.$$

Problem 2. *(5 points.)* Find the most general antiderivative of the following function

$$f(x) = \frac{\sqrt{x} - x^4 + x^5\sin(x)}{x^5}.$$

Midterm 2

Problem 1. i) *(5 points.)* Evaluate the following limit

$$\lim_{x\to\pi^-} \left(\tan\left(\frac{x}{2}\right)\right)^{\cos\left(\frac{x}{2}\right)}.$$

Solution. One has

$$\begin{aligned}
\lim_{x\to\pi^-} \left(\tan\left(\frac{x}{2}\right)\right)^{\cos\left(\frac{x}{2}\right)} &= \lim_{x\to\pi^-} e^{\cos\left(\frac{x}{2}\right)\ln\left(\tan\left(\frac{x}{2}\right)\right)} = \lim_{x\to\pi^-} e^{\frac{\ln\left(\tan\left(\frac{x}{2}\right)\right)}{\left(\cos\left(\frac{x}{2}\right)\right)^{-1}}}\\
&= \lim_{x\to\pi^-} e^{\frac{\frac{1}{2\tan\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right)}}{\frac{\sin\left(\frac{x}{2}\right)}{2\cos^2\left(\frac{x}{2}\right)}}} = \lim_{x\to\pi^-} e^{\frac{\cos^2\left(\frac{x}{2}\right)}{\tan\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right)}}\\
&= \lim_{x\to\pi^-} e^{\frac{\cos\left(\frac{x}{2}\right)}{\sin^2\left(\frac{x}{2}\right)}} = e^0 = 1.
\end{aligned}$$

□

ii) *(5 points.)* Find the horizontal asymptotes to the graph of the following function

$$f(x) = \left(\cos\left(\frac{2}{x}\right)\right)^{x^2}.$$

Solution. One computes

$$\lim_{x\to\infty}\left(\cos\left(\frac{2}{x}\right)\right)^{x^2} = \lim_{x\to\infty} e^{x^2\ln\left(\cos\left(\frac{2}{x}\right)\right)} = \lim_{x\to\infty} e^{\frac{\ln\left(\cos\left(\frac{2}{x}\right)\right)}{x^{-2}}} = \lim_{x\to\infty} e^{\frac{\frac{2\sin\left(\frac{2}{x}\right)}{x^2\cos\left(\frac{2}{x}\right)}}{\frac{-2}{x^3}}}$$

$$= \lim_{x\to\infty} e^{-\frac{x\sin\left(\frac{2}{x}\right)}{\cos\left(\frac{2}{x}\right)}} = \lim_{x\to\infty} e^{-\frac{1}{\cos\left(\frac{2}{x}\right)}\cdot 2\cdot\frac{\sin\left(\frac{2}{x}\right)}{\frac{2}{x}}} = e^{-2}.$$

Moreover, the function $f(x)$ is even, hence $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = e^{-2}$. We deduce that there is only one horizontal asymptote, namely $y = e^{-2}$. □

Problem 2. i)*(6 points.)* Given

$$f'(x) = \frac{(2-x)(x^2+3)(x+4)}{x^3(x-1)^4},$$

find the critical points of the function $f(x)$. Decide which critical points give a local maximum value, and which give a local minimum value.

Solution. Note that the factors (x^2+3) and $(x-1)^4$ are always non-negative, and hence do not affect the study of the sign of $f'(x)$. The critical values of $f(x)$ are $-4, 0, 1, 2$. By the first derivative test, $x = -4$ and $x = 2$ give a local maximum, $x = 0$ gives a local minimum, while $x = 1$ does not give a local extreme. □

ii) *(6 points.)* Given

$$f'(x) = \frac{x^3(3x^2-5)}{15},$$

find the intervals of concavity and the inflection points of the function $f(x)$.

Solution. Note that

$$f'(x) = \frac{x^5}{5} - \frac{x^3}{3}.$$

One computes

$$f''(x) = x^4 - x^2 = x^2(x-1)(x+1).$$

From the study of the sign of $f''(x)$, the function $f(x)$ is concave upward on the intervals $x < -1$ and $x > 1$, and is concave downward on the interval $-1 < x < 1$. It follows that the inflection points are at $x = -1$ and $x = 1$. □

Problem 3. i) *(5 points.)* Find the local and absolute extreme values of the following function

$$f(x) = 2x^2 - x^4$$

on the interval $[-2, 2]$. Decide which local extreme values are a local minimum or a local maximum.

Solution. Note that the function $f(x)$ is even, so the graph of $f(x)$ is symmetric with respect to the y-axis. One computes

$$f'(x) = 4x - 4x^3 = 4x(1-x)(1+x).$$

Hence, local or absolute values may occur at $x \in \{-2, -1, 0, 1, 2\}$, that is, at the critical points, or at the endpoints. Since $f(0) = 0$, $f(1) = f(-1) = 1$, and $f(-2) = f(2) = -8$, one has that the absolute maximum is at $x = 1$ and $x = -1$, and the absolute minimum is at $x = 2$ and $x = -2$. From the first derivative test, one deduces that $x = 0$ gives a local minimum. □

ii) *(5 points.)* Given

$$f'(x) = \sec(x)(\sec(x) + e^x \cos(x))$$

and $f(\frac{\pi}{4}) = 1$, find $f(x)$.

Solution. One computes

$$f(x) = \int f'(x)dx = \int (\sec^2(x) + e^x)dx = \tan(x) + e^x + C.$$

Since

$$1 = f\left(\frac{\pi}{4}\right) = 1 + e^{\frac{\pi}{4}} + C,$$

we deduce $C = -e^{\frac{\pi}{4}}$. Hence, we have $f(x) = \tan(x) + e^x - e^{\frac{\pi}{4}}$. □

Problem 4. i) *(6 points.)* Find the general antiderivative of the following function

$$f(x) = (2-x)\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right).$$

Solution. We have

$$\begin{aligned}
\int (2-x)\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx &= \int \left(2\sqrt{x} + \frac{2}{\sqrt{x}} - x\sqrt{x} - \frac{x}{\sqrt{x}}\right) dx \\
&= \int \left(2x^{\frac{1}{2}} + 2x^{-\frac{1}{2}} - x^{\frac{3}{2}} - x^{\frac{1}{2}}\right) dx \\
&= \frac{4}{3}x^{\frac{3}{2}} + 4\sqrt{x} - \frac{2}{5}x^{\frac{5}{2}} - \frac{2}{3}x^{\frac{3}{2}} + C.
\end{aligned}$$

□

ii) *(6 points.)* Find the general antiderivative of the following function

$$f(x) = \frac{7\sqrt{x} - \sqrt[3]{x} \cdot 3^x - x^{-\frac{2}{3}}}{\sqrt[3]{x}}.$$

Solution. We have

$$\begin{aligned}\int \frac{7\sqrt{x} - \sqrt[3]{x}\cdot 3^x - x^{-\frac{2}{3}}}{\sqrt[3]{x}}dx &= 7\int x^{\frac{1}{6}}dx - \int 3^x dx - \int \frac{1}{x}dx \\ &= 6x^{\frac{7}{6}} - \frac{3^x}{\ln(3)} - \ln|x| + C.\end{aligned}$$

□

Problem 5. i) *(5 points.)* Evaluate the following integral

$$\int_{-\frac{3}{2}}^{\frac{7}{2}} \lfloor x \rfloor dx.$$

Hint: sketch the graph of the function $f(x) = \lfloor x \rfloor$.

Solution. We have

$$\begin{aligned}\int_{-\frac{3}{2}}^{\frac{7}{2}} \lfloor x \rfloor dx &= \int_{-1.5}^{-1}(-2)dx + \int_{-1}^{0}(-1)dx + \int_{0}^{1} 0\cdot dx + \int_{1}^{2} 1\cdot dx + \int_{2}^{3} 2\cdot dx + \int_{3}^{3.5} 3\cdot dx \\ &= \left(-2\cdot\frac{1}{2}\right) + (-1) + 0 + 1 + 2 + \left(3\cdot\frac{1}{2}\right) = \frac{5}{2}.\end{aligned}$$

□

ii) *(5 points.)* Evaluate the following integral

$$\int_{-\frac{\pi}{6}}^{\frac{5\pi}{4}} |\sin(x)|dx.$$

Solution. We have

$$\begin{aligned}\int_{-\frac{\pi}{6}}^{\frac{5\pi}{4}} |\sin(x)|dx &= \int_{-\frac{\pi}{6}}^{0}(-\sin(x))dx + \int_{0}^{\pi}\sin(x)dx + \int_{\pi}^{\frac{5\pi}{4}}(-\sin(x))dx \\ &= [\cos(x)]_{-\frac{\pi}{6}}^{0} + [-\cos(x)]_0^{\pi} + [\cos(x)]_{\pi}^{\frac{5\pi}{4}} \\ &= \cos(0) - \cos\left(-\frac{\pi}{6}\right) + (-\cos(\pi) - (-\cos(0))) + \cos\left(\frac{5\pi}{4}\right) - \cos(\pi) \\ &= 1 - \frac{\sqrt{3}}{2} + 2 + \left(-\frac{\sqrt{2}}{2} + 1\right) = 4 - \frac{\sqrt{3}+\sqrt{2}}{2}.\end{aligned}$$

□

Problem 6. *(6 points.)* Given

$$f(x) := \begin{cases} |x| - 1, & \text{for } x \leq 1 \\ e^x - e, & \text{for } 1 < x < 2 \\ -\frac{1}{x^2}, & \text{for } x \geq 2 \end{cases},$$

evaluate the following integral

$$\int_{-1}^{3} f(x)dx.$$

Hint: sketch the graph of the function $f(x)$.

Solution. We have

$$\begin{aligned} \int_{-1}^{3} f(x)dx &= \int_{-1}^{0} (-x-1)dx + \int_{0}^{1} (x-1)dx \int_{1}^{2} (e^x - e)dx + \int_{2}^{3} \left(-\frac{1}{x^2}\right) dx \\ &= \left[-\frac{x^2}{2} - x\right]_{-1}^{0} + \left[\frac{x^2}{2} - x\right]_{0}^{1} + [e^x - e \cdot x]_1^2 + \left[\frac{1}{x}\right]_2^3 \\ &= \frac{1}{2} - 1 + \frac{1}{2} - 1 + e^2 - 2e - \frac{1}{6} = -\frac{7}{6} + e^2 - 2e. \end{aligned}$$

□

ii) *(2 bonus points.)* Evaluate the following limit by first recognizing the sum as a Riemann sum for a function defined on $[0, 1]$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\sqrt{1 - \left(\frac{i}{n}\right)^2}}.$$

Solution. We have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\sqrt{1 - \left(\frac{i}{n}\right)^2}} = \int_0^1 \frac{1}{\sqrt{1-x^2}} dx = \arcsin(1) - \arcsin(0) = \frac{\pi}{2}.$$

□

Week 15 Quiz

Problem. Compute the following integrals

$$(i) \int e^{3t}(t^2 + t)dt,$$

$$(ii) \int \frac{8 - 5x}{(x-1)(2-x)} dx.$$

Final Exam

Problem 1. *(9 points.)* Consider the following function

$$f(x) = \ln(x) - \frac{x^2}{6}.$$

(1) Find the domain of $f(x)$.
(2) Find the interval(s) on the real line where $f(x)$ is increasing and decreasing.
(3) Find the x-value(s), if any, where $f(x)$ has zero slope.
(4) Find the point(s) of inflection, if any, for $f(x)$ and regions of the real line where the function is concave up and down.
(5) Based on the results from above, sketch the graph of $f(x)$, and correctly represent increasing and decreasing regions, and concavity. Are there any vertical asymptotes to the graph of $f(x)$?

Solution. The domain of $f(x)$ is the interval $(0, \infty)$. One has

$$f'(x) = \frac{1}{x} - \frac{x}{3} = \frac{3 - x^2}{3x}.$$

One finds that $f'(x)$ is positive on the interval $(0, \sqrt{3})$, and negative on the interval $(\sqrt{3}, \infty)$. It follows that $f(x)$ is increasing on $(0, \sqrt{3})$ and decreasing on $(\sqrt{3}, \infty)$. Moreover, $f(x)$ has zero slope at $x = \sqrt{3}$. One has

$$f''(x) = -\frac{1}{x^2} - \frac{1}{3} = -\frac{3 + 2x^2}{3x^2}.$$

Since $f''(x)$ is always negative, one deduces that $f(x)$ is concave down on its domain. There are no inflection points. Finally, there is a global maximum at $x = \sqrt{3}$, and there is a vertical asymptote at $x = 0$. □

Problem 2. i) *(4 points.)* Suppose the following function

$$f(t) = t \sin(t^2)$$

represents the velocity of an object for t in the range $\left[0, \sqrt{\frac{\pi}{6}}\right]$.

(1) Write down the correct expression for the distance traveled starting from time $t = 0$ to time $t = \sqrt{\frac{\pi}{6}}$.
(2) Compute the exact value of the expression from (1).

Solution. The distance is

$$\int_0^{\sqrt{\frac{\pi}{6}}} t \sin(t^2) dt = \frac{1}{2} \left[-\cos(t^2)\right]_0^{\sqrt{\frac{\pi}{6}}} = \frac{1}{2} \left(1 - \frac{\sqrt{3}}{2}\right).$$

□

ii) *(5 points.)* Find the horizontal asymptotes to the graph of the following function

$$f(x) = \sqrt{\frac{3x^2(1-3x)(1+3x)}{(x^2-1)(4-3x^2)}}.$$

Solution. To compute the horizontal asymptotes, one has to compute the limit of $f(x)$ as $x \to \pm\infty$. We have

$$\lim_{x\to\infty} f(x) = \sqrt{\frac{-27x^4}{-3x^4}} = 3 = \lim_{x\to-\infty} f(x).$$

Hence, there is one horizontal asymptote, namely $y = 3$. □

Problem 3. *(15 points.)* Find the derivative of the following functions

$$\begin{aligned} f(x) &= \sqrt[3]{x}\sin(\cos(1-x^2)), \\ g(x) &= \int_0^{\cos(x)} \frac{e^y}{\sqrt[3]{y}+2}dy, \\ h(x) &= (\ln(3x))^{\sin(x)}. \end{aligned}$$

Solution. One has

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}\sin(\cos(1-x^2)) + \sqrt[3]{x}\cos(\cos(1-x^2))\cdot(-\sin(1-x^2))(-2x).$$

To compute $g'(x)$, let

$$L(x) := \int_0^x \frac{e^y}{\sqrt[3]{y}+2}dy.$$

Note that $L'(x) = \frac{e^x}{\sqrt[3]{x}+2}$. We can rewrite $g(x)$ as $g(x) = L(\cos(x))$. We deduce

$$g'(x) = L'(\cos(x))\cdot\cos'(x) = \frac{e^{\cos(x)}}{\sqrt[3]{\cos(x)}+2}\cdot(-\sin(x)).$$

Finally, to compute $h'(x)$, we use the method of logarithmic differentiation. We have

$$\ln(h(x)) = \sin(x)\cdot\ln(\ln(3x)).$$

Differentiating, we have

$$\frac{1}{h(x)}\cdot h'(x) = \cos(x)\cdot\ln(\ln(3x)) + \sin(x)\cdot\frac{1}{\ln(3x)}\cdot\frac{1}{3x}3.$$

Finally, solving for $h'(x)$, we have

$$h'(x) = h(x)\cdot\left(\cos(x)\cdot\ln(\ln(3x)) + \frac{\sin(x)}{x\ln(3x)}\right).$$

□

Problem 4. i) *(4 points.)* Evaluate the following limit

$$\lim_{x\to 0} \frac{\ln(3x+1)-3x}{e^{-x}-1+x}.$$

Solution. Applying l'Hopital twice, one has

$$\lim_{x\to 0} \frac{\ln(3x+1)-3x}{e^{-x}-1+x} = \lim_{x\to 0} \frac{\frac{3}{3x+1}-3}{-e^{-x}+1} = \lim_{x\to 0} \frac{-\frac{9}{(3x+1)^2}}{e^{-x}} = -9.$$

□

ii) *(8 points.)* Evaluate the following integral

$$\int \frac{x^2-1}{x^3+x^2+x} dx.$$

Solution. By the method of partial fractions decomposition, one has

$$\int \frac{x^2-1}{x^3+x^2+x} dx = \int \frac{-1}{x} dx + \int \frac{2x+1}{x^2+x+1} dx = -\ln|x| + \ln(x^2+x+1) + C.$$

□

Problem 5. i) *(5 points.)* Evaluate the following integral

$$\int_0^{\frac{1}{2}} (1-t^2)\sin(\pi t)dt.$$

Solution. Integrating by parts twice, one has

$$\begin{aligned} \int (1-t^2)\sin(\pi t)dt &= -\frac{\cos(\pi t)}{\pi}(1-t^2) - \frac{2}{\pi}\int \cos(\pi t)t dt \\ &= -\frac{\cos(\pi t)}{\pi}(1-t^2) - \frac{2}{\pi^2}\sin(\pi t)t - \frac{2}{\pi^3}\cos(\pi t) + C. \end{aligned}$$

Hence, the definite integral is

$$\int_0^{\frac{1}{2}} (1-t^2)\sin(\pi t)dt = \left[-\frac{\cos(\pi t)}{\pi}(1-t^2) - \frac{2}{\pi^2}\sin(\pi t)t - \frac{2}{\pi^3}\cos(\pi t)\right]_0^{\frac{1}{2}} = \frac{1}{\pi} - \frac{1}{\pi^2} + \frac{2}{\pi^3}.$$

□

ii) *(5 points.)* Evaluate the following integral or show that it is divergent

$$\int_2^{\infty} \frac{x}{\sqrt{x^2-1}} dx.$$

Solution. We have
$$\int_2^{\infty} \frac{x}{\sqrt{x^2-1}}dx = \lim_{b\to\infty} \frac{1}{2}\left[2\sqrt{x^2-1}\right]_2^b = \lim_{b\to\infty}(\sqrt{b^2-1}-\sqrt{3}).$$
The limit is ∞, hence the integral diverges. □

Problem 6. i) *(5 points.)* Find the volume of the solid obtained by rotating about the x-axis the region bounded by the curves $x=2$, $y=x$, and $y^2=x+2$.

Solution. The volume is
$$\pi\int_0^2 (x+2-x^2)dx = \pi\left[\frac{x^2}{2}+2x-\frac{x^3}{3}\right]_0^2 = \frac{10\pi}{3}.$$
□

ii) *(5 bonus points.)*. Evaluate the following limit by first recognizing the sum as a Riemann sum for a function defined on $[0,1]$
$$\lim_{n\to\infty}\frac{\pi}{n}\sum_{i=1}^{n}\frac{1}{1+\left(\frac{i}{n}\right)^2}.$$
Describe a solid of revolution whose volume is equal to the above limit.

Solution. One has
$$\lim_{n\to\infty}\frac{\pi}{n}\sum_{i=1}^{n}\frac{1}{1+\left(\frac{i}{n}\right)^2} = \pi\int_0^1\frac{1}{1+x^2}dx = \pi[\arctan(x)]_0^1 = \frac{\pi^2}{4}.$$
A possible solid of revolution with volume equal to the above limit is the solid obtained by revolving about the x-axis the region bounded by the curves $y=0$, $x=0$, $x=1$, $y^2=\frac{1}{1+x^2}$. A simpler solution is the cylinder of height 1 and radius of the base equal to $\sqrt{\pi}/2$. □