DOUBLE POINTS OF PLANE MODELS IN $\overline{\mathcal{M}}_{6,1}$

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Abstract. The aim of this paper is to compute the class of the closure of the effective divisor $\mathfrak{D}^2_6$ in $\mathcal{M}_{6,1}$ given by pointed curves $[C,p]$ with a sextic plane model mapping p to a double point. Such a divisor generates an extremal ray in the pseudoeffective cone of $\overline{\mathcal{M}}_{6,1}$ as shown by Jensen. A general result on some families of linear series with adjusted Brill-Noether number 0 or -1 is introduced to complete the computation.

The birational geometry of an algebraic variety is encoded in its cone of effective divisors. Nowadays a major problem is to determine the effective cone of moduli spaces of curves.

Let $\mathcal{GP}^1_4$ be the Gieseker-Petri divisor in $\mathcal{M}_6$ given by curves with a $\mathfrak{g}^1_4$ violating the Petri condition. The class

$$\left[\overline{\mathcal{GP}}^1_4\right] = 94\lambda - 12\delta_0 - 50\delta_1 - 78\delta_2 - 88\delta_3 \in \mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_6)$$

is computed in [EH87] where classes of Brill-Noether divisors and Gieseker-Petri divisors are determined for arbitrary genera in order to prove that $\overline{\mathcal{M}}_g$ is of general type for $g \geq 24$.

Now let $\mathfrak{D}^2_d$ be the divisor in $\mathcal{M}_{g,1}$ defined as the locus of smooth pointed curves $[C,p]$ with a net $\mathfrak{g}^2_d$ of Brill-Noether number 0 mapping p to a double point. That is

$$\mathfrak{D}^2_d := \left\{ [C,p] \in \mathcal{M}_{g,1} \mid \exists\, l \in G^2_d(C) \text{ with } l(-p-x) \in G^1_{d-2}(C) \text{ where } x \in C, x \neq p \right\}$$

for values of g, d such that $g = 3(g-d+2)$. Recently Jensen has shown that $\overline{\mathfrak{D}}^2_6$ and the pull-back of $\overline{\mathcal{GP}}^1_4$ to $\overline{\mathcal{M}}_{6,1}$ generate extremal rays of the pseudoeffective cone of $\overline{\mathcal{M}}_{6,1}$ (see [Jen10]). Our aim is to prove the following theorem.

Theorem 1. *The class of the divisor $\overline{\mathfrak{D}}^2_6 \subset \overline{\mathcal{M}}_{6,1}$ is*

$$\left[\overline{\mathfrak{D}}^2_6\right] = 62\lambda + 4\psi - 8\delta_0 - 30\delta_1 - 52\delta_2 - 60\delta_3 - 54\delta_4 - 34\delta_5 \in \mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{6,1}).$$

A mix of a Porteous-type argument, the method of test curves and a pull-back to rational pointed curves will lead to the result. Following a method described in [Kho07], we realize $\overline{\mathfrak{D}}^2_d$ in $\mathcal{M}^{\mathrm{irr}}_{g,1}$ as the push-forward of a degeneracy locus of a map of vector bundles over $\mathcal{G}^2_d(\mathcal{M}^{\mathrm{irr}}_{g,1})$. This will give us the coefficients of λ, ψ and δ_0 for the class of $\overline{\mathfrak{D}}^2_d$ in general. Intersecting $\overline{\mathfrak{D}}^2_d$ with carefully chosen one-dimensional families of curves will produce relations to determine the coefficients of δ_1 and δ_{g-1}. Finally in the case $g = 6$ we will get enough relations to find the other coefficients by pulling-back to the moduli space of stable pointed rational curves in the spirit of [EH87, §3].

To complete our computation we obtain a general result on some families of linear series on pointed curves with adjusted Brill-Noether number $\rho = 0$ that essentially excludes further ramifications on such families.

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Theorem 2. *Let (C,y) be a general pointed curve of genus $g>1$. Let l be a $\mathfrak{g}^r_d$ on C with $r\geq 2$ and adjusted Brill-Noether number $\rho(C,y)=0$. Denote by $(a_0,a_1,\dots,a_r)$ the vanishing sequence of l at y. Then $l(-a_iy)$ is base-point free for $i=0,\dots,r-1$.*

For instance if C is a general curve of genus 4 and $l\in G^2_5(C)$ has vanishing sequence $(0,1,3)$ at a general point p in C, then $l(-p)$ is base-point free.

Using the irreducibility of the families of linear series with adjusted Brill-Noether number -1 ([EH89]), we get a similar statement for an arbitrary point on the general curve in such families.

Theorem 3. *Let C be a general curve of genus $g>2$. Let l be a $\mathfrak{g}^r_d$ on C with $r\geq 2$ and adjusted Brill-Noether number $\rho(C,y)=-1$ at an arbitrary point y. Denote by $(a_0,a_1,\dots,a_r)$ the vanishing sequence of l at y. Then $l(-a_1y)$ is base-point free.*

As a verification of Thm. 1, let us note that the class of $\overline{\mathfrak{D}}^2_6$ is not a linear combination of the class of the Gieseker-Petri divisor $\mathcal{GP}^1_4$ and the class of the divisor $\mathcal{W}$ of Weierstrass points computed in [Cuk89]

$$[\mathcal{W}] = -\lambda+21\psi-15\delta_1-10\delta_2-6\delta_3-3\delta_4-\delta_5\in \mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{6,1}).$$

After briefly recalling in the next section some basic results about limit linear series and enumerative geometry on the general curve, we prove Thm. 2 and Thm. 3 in section 2. Finally in section 3 we prove a general version of Thm. 1.

1. Limit linear series and enumerative geometry

We use throughout Eisenbud and Harris's theory of limit linear series (see [EH86]). Let us recall some basic definitions and results.

1.1. **Linear series on pointed curves.** Let C be a complex smooth projective curve of genus g and $l=(\mathscr{L},V)$ a linear series of type $\mathfrak{g}^r_d$ on C, that is $\mathscr{L}\in\mathrm{Pic}^d(C)$ and $V\subset H^0(\mathscr{L})$ is a subspace of vector-space dimension $r+1$. The *vanishing sequence* $a^l(p): 0\leq a_0<\cdots<a_r\leq d$ of l at a point $p\in C$ is defined as the sequence of distinct order of vanishing of sections in V at p, and the *ramification sequence* $\alpha^l(p): 0\leq\alpha_0\leq\cdots\leq\alpha_r\leq d-r$ as $\alpha_i:=a_i-i$, for $i=0,\dots,r$. The *weight* $w^l(p)$ will be the sum of the α_i's.

Given an n-pointed curve $(C,p_1,\dots,p_n)$ of genus g and l a $\mathfrak{g}^r_d$ on C, the *adjusted Brill-Noether number* is

$$\rho(C,p_1,\dots p_n)=\rho(g,r,d,\alpha^l(p_1),\dots,\alpha^l(p_n)):=g-(r+1)(g-d+r)-\sum_{i,j}\alpha^l_j(p_i).$$

1.2. **Counting linear series on the general curve.** Let C be a general curve of genus $g>0$ and consider r,d such that $\rho(g,r,d)=0$. Then by Brill-Noether theory, the curve C admits only a finite number of $\mathfrak{g}^r_d$'s computed by the *Castelnuovo number*

$$N_{g,r,d}:=g!\prod_{i=0}^{r}\frac{i!}{(g-d+r+i)!}.$$

Furthermore let (C,p) be a general pointed curve of genus $g>0$ and let $\overline{\alpha}=(\alpha_0,\dots,\alpha_r)$ be a Schubert index of type r,d (that is $0\leq\alpha_0\leq\cdots\leq\alpha_r\leq d-r$) such that $\rho(g,r,d,\overline{\alpha})=0$. Then by [EH87, Prop. 1.2], the curve C admits a $\mathfrak{g}^r_d$ with ramification sequence $\overline{\alpha}$ at the point p if and only if $\alpha_0+g-d+r\geq 0$. When such linear series exist, there is a finite number of them counted by the following formula

$$N_{g,r,d,\overline{\alpha}}:=g!\frac{\prod_{i<j}(\alpha_j-\alpha_i+j-i)}{\prod_{i=0}^{r}(g-d+r+\alpha_i+i)!}.$$

1.3. **Limit linear series.** For a curve of compact type $C = Y_1 \cup \cdots \cup Y_s$ of arithmetic genus g with nodes at the points $\{p_{ij}\}_{ij}$, let $\{l_{Y_1}, \ldots l_{Y_s}\}$ be a limit linear series $\mathfrak{g}^r_d$ on C. Let $\{q_{ik}\}_k$ be smooth points on Y_i, $i = 1, \ldots, s$. In [EH86] a moduli space of such limit series is constructed as a disjoint union of schemes on which the vanishing sequences of the aspects l_{Y_i}'s at the nodes are specified. A key property is the additivity of the adjusted Brill-Noether number, that is

$$\rho(g, r, d, \{\alpha^{l_{Y_i}}(q_{ik})\}_{ik}) \geq \sum_i \rho(Y_i, \{p_{ij}\}_j, \{q_{ik}\}_k).$$

The smoothing result [EH86, Cor. 3.7] assures the smoothability of dimensionally proper limit series. The following facts ease the computations. The adjusted Brill-Noether number for any $\mathfrak{g}^r_d$ on one-pointed elliptic curves or on n-pointed rational curves is nonnegative. For a general curve C of arbitrary genus g, one has $\rho(C, p) \geq 0$ for p general in C and $\rho(C, y) \geq -1$ for any $y \in C$ (see [EH89]).

2. Ramifications on some families of linear series with $\rho = 0$ or -1

Here we prove Thm. 2. The result will be repeatedly used in the next section.

Proof of Thm. 2. Clearly it is enough to prove the statement for $i = r - 1$. We proceed by contradiction. Suppose that for (C, y) a general pointed curve of genus g, there exists $x \in C$ such that $h^0(l(-a_{r-1}y - x)) \geq 2$, for some l a $\mathfrak{g}^r_d$ with $\rho(C, y) = 0$. Let us degenerate C to a transversal union $C_1 \cup_{y_1} E_1$, where C_1 has genus $g - 1$ and E_1 is an elliptic curve. Since y is a general point, we can assume $y \in E_1$ and $y - y_1$ not to be a $d!$-torsion point in $\mathrm{Pic}^0(E_1)$. Let $\{l_{C_1}, l_{E_1}\}$ be a limit $\mathfrak{g}^r_d$ on $C_1 \cup_{y_1} E_1$ such that $a^{l_{E_1}}(y) = (a_0, a_1, \ldots, a_r)$. Denote by $(\alpha_0, \ldots, \alpha_r)$ the corresponding ramification sequence. We have that $\rho(C_1, y_1) = \rho(E_1, y, y_1) = 0$, hence $w^{l_{C_1}}(y_1) = r + \rho$, where $\rho = \rho(g, r, d)$. Denote by $(b^1_0, b^1_1, \ldots, b^1_r)$ the vanishing sequence of l_{C_1} at y_1 and by $(\beta^1_0, \beta^1_1, \ldots, \beta^1_r)$ the corresponding ramification sequence.

Suppose x specializes to E_1. Then $b^1_r \geq a_r + 1$, $b^1_{r-1} \geq a_{r-1} + 1$ and we cannot have both equalities, since $y - y_1$ is not in $\mathrm{Pic}^0(E_1)[d!]$ (see for instance [Far00, Prop. 4.1]). Moreover, as usually $b^1_k \geq a_k$ for $0 \leq k \leq r - 2$, and again among these inequalities there cannot be more than one equality. We deduce

$$w^{l_{C_1}}(y_1) \geq w^{l_{E_1}}(y) + 3 + r - 2 > w^{l_{E_1}}(y) + r = r + \rho$$

hence a contradiction. We have supposed that $h^0(l(-a_{r-1}y - x)) \geq 2$. Then this pencil degenerates to $l_{E_1}(-a_{r-1}y)$ and to a compatible sub-pencil l'_{C_1} of $l_{C_1}(-x)$. We claim that

$$h^0\left(l_{C_1}\left(-b^1_{r-1}y_1 - x\right)\right) \geq 2.$$

Suppose this is not the case. Then we have $a^{l_{C_1}(-x)}(y_1) \leq (b^1_0, \ldots, b^1_{r-2}, b^1_r)$, hence $b^1_r \geq a_r$, $b^1_{r-2} \geq a_{r-1}$ and $b^1_k \geq a_k$, for $0 \leq k \leq r - 3$. Among these, we cannot have more than one equality, plus $\beta^1_{r-2} \geq \alpha_{r-1} + 1$ and $\beta^1_{r-1} \geq \beta^1_{r-2} > \alpha_{r-1} \geq \alpha_{r-2}$, hence

$$w^{l_{C_1}}(y_1) \geq w^{l_{E_1}}(y) + 1 + r - 1 + \beta^1_{r-1} - \alpha_{r-2} > r + \rho$$

a contradiction.

From our assumptions, we have deduced that for (C_1, y_1) a general pointed curve of genus $g - 1$, there exist l_{C_1} a $\mathfrak{g}^r_d$ and $x \in C_1$ such that $\rho(C_1, y_1) = 0$ and $h^0(l_{C_1}(-b^1_{r-1}y_1 - x)) \geq 2$, where b^1_{r-1} is as before.

Then we apply the following recursive argument. At the step i, we degenerate the pointed curve (C_i, y_i) of genus $g - i$ to a transversal union $C_{i+1} \cup_{y_{i+1}} E_{i+1}$, where C_{i+1} is a curve of genus $g - i - 1$ and E_{i+1} is an elliptic curve, such that $y_i \in E_{i+1}$. Let $\{l_{C_{i+1}}, l_{E_{i+1}}\}$ be a limit $\mathfrak{g}^r_d$ on $C_{i+1} \cup_{y_{i+1}} E_{i+1}$ such that

$a^{l_{E_{i+1}}}(y_i) = (b_0^i, b_1^i, \dots, b_r^i)$. From $\rho(C_{i+1}, y_{i+1}) = \rho(E_{i+1}, y_i, y_{i+1}) = 0$, we compute that $w^{l_{C_{i+1}}}(y_{i+1}) = (i+1)r + \rho$. Denote by $(b_0^{i+1}, b_1^{i+1}, \dots, b_r^{i+1})$ the vanishing sequence of $l_{C_{i+1}}$ at y_{i+1}. As before we arrive to a contradiction if $x \in E_{i+1}$, and we deduce

$$h^0\left(l_{C_{i+1}}\left(-b_{r-1}^{i+1} y_{i+1} - x\right)\right) \geq 2.$$

At the step $g-2$, our degeneration produces two elliptic curves $C_{g-1} \cup_{y_{g-1}} E_{g-1}$, with $y_{g-2} \in E_{g-1}$. Our assumptions yield the existence of $x \in C_{g-1}$ such that

$$h^0(l_{C_{g-1}}(-b_{r-1}^{g-1} y_{g-1} - x)) \geq 2.$$

We compute $w^{l_{C_{i+1}}}(y_{g-1}) = (g-1)r + \rho$. By the numerical hypothesis, we see that $(g-1)r + \rho = (d-r-1)(r+1) + 1$, hence the vanishing sequence of $l_{C_{g-1}}$ at y_{g-1} has to be $(d-r-1, \dots, d-3, d-2, d)$. Whence the contradiction. □

The following proves the similar result for some families of linear series with Brill-Noether number -1.

Proof of Thm 3. The statement says that for every $y \in C$ such that $\rho(C, y) = -1$ for some l a $\mathfrak{g}^r_d$, and for every $x \in C$, we have that $h^0(l(-a_1 y - x)) \leq r - 1$. This is a closed condition and, using the irreducibility of the divisor $\mathcal{D}$ of pointed curves admitting a linear series $\mathfrak{g}^r_d$ with adjusted Brill-Noether number -1, it is enough to prove it for $[C, y]$ general in $\mathcal{D}$.

We proceed by contradiction. Suppose for $[C, y]$ general in $\mathcal{D}$ there exists $x \in C$ such that $h^0(l(-a_1 y - x)) \geq r$ for some l a $\mathfrak{g}^r_d$ with $\rho(C, y) = -1$. Let us degenerate C to a transversal union $C_1 \cup_{y_1} E_1$ where C_1 is a general curve of genus $g-1$ and E_1 is an elliptic curve. Since y is a general point, we can assume $y \in E_1$. Let $\{l_{C_1}, l_{E_1}\}$ be a limit $\mathfrak{g}^r_d$ on $C_1 \cup_{y_1} E_1$ such that $a^{l_{E_1}}(y) = (a_0, a_1, \dots, a_r)$. Then $\rho(E_1, y, y_1) \leq -1$ and $\rho(C_1, y_1) = 0$, hence $w^{l_{C_1}}(y_1) = r + \rho$ (see also [Far09, Proof of Thm. 4.6]). Let $(b_0^1, b_1^1, \dots, b_r^1)$ be the vanishing sequence of l_{C_1} at y_1 and $(\beta_0^1, \beta_1^1, \dots, \beta_r^1)$ the corresponding ramification sequence.

The point x has to specialize to C_1. Indeed suppose $x \in E_1$. Then $b_k^1 \geq a_k + 1$ for $k \geq 1$. This implies $w^{l_{C_1}}(y_1) \geq w^{l_{E_1}}(y) + r > \rho + r$, hence a contradiction. Then $x \in C_1$, and $l(-a_1 y - x)$ degenerates to $l_{E_1}(-a_1 y)$ and to a compatible system $l'_{C_1} := l_{C_1}(-x)$. We claim that

$$h^0\left(l_{C_1}\left(-b_{r-1}^1 y_1 - x\right)\right) \geq 2.$$

Suppose this is not the case. Then we have $a^{l'_{C_1}}(y_1) \leq (b_0^1, \dots, b_{r-2}^1, b_r^1)$ and so $b_r^1 \geq a_r$, and $b_k^1 \geq a_{k+1}$ for $0 \leq k \leq r-2$. Then $\beta_k^1 \geq \alpha_{k+1} + 1$ for $k \leq r-2$, and summing up we receive

$$w^{l_{C_1}}(y_1) \geq w^{l_{E_1}}(y) + r - 1 + \beta_{r-1}^1 - \alpha_0.$$

Clearly $\beta_{r-1}^1 \geq \beta_{r-2}^1 > \alpha_{r-1} \geq \alpha_0$. Hence $w^{l_{C_1}}(y_1) > \rho + r$, a contradiction.

All in all from our assumptions we have deduced that for a general pointed curve (C_1, y_1) of genus $g-1$, there exist l_{C_1} a $\mathfrak{g}^r_d$ and $x \in C_1$ such that $\rho(C_1, y_1) = 0$ and $h^0(l_{C_1}(-b_{r-1}^1 y_1 - x)) \geq 2$, where b_{r-1}^1 is as before. This contradicts Thm. 2, hence we receive the statement. □

3. The divisor $\mathfrak{D}^2_d$

Remember that $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,1})$ is generated by the Hodge class λ, the cotangent class ψ corresponding to the marked point, and the boundary classes $\delta_0, \dots \delta_{g-1}$ defined as follows. The class δ_0 is the class of the closure of the locus of pointed irreducible nodal curves, and the class δ_i is the class of the closure of the locus of pointed curves $[C_i \cup C_{g-i}, p]$ where C_i and C_{g-i} are smooth curves respectively of genus

i and $g-i$ meeting transversally in one point, and p is a smooth point in C_i, for $i = 1, \dots, g-1$. In this section we prove the following theorem.

Theorem 4. *Let $g = 3s$ and $d = 2s+2$ for $s \geq 1$. The class of the divisor $\overline{\mathfrak{D}}^2_d$ in $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,1})$ is*

$$\left[\overline{\mathfrak{D}}^2_d\right] = a\lambda + c\psi - \sum_{i=0}^{g-1} b_i \delta_i$$

where

$$\begin{aligned} a &= \frac{48s^4 + 80s^3 - 16s^2 - 64s + 24}{(3s-1)(3s-2)(s+3)} N_{g,2,d} \\ c &= \frac{2s(s-1)}{3s-1} N_{g,2,d} \\ b_0 &= \frac{24s^4 + 23s^3 - 18s^2 - 11s + 6}{3(3s-1)(3s-2)(s+3)} N_{g,2,d} \\ b_1 &= \frac{14s^3 + 6s^2 - 8s}{(3s-2)(s+3)} N_{g,2,d} \\ b_{g-1} &= \frac{48s^4 + 12s^3 - 56s^2 + 20s}{(3s-1)(3s-2)(s+3)} N_{g,2,d}. \end{aligned}$$

Moreover for $g = 6$ and for $i = 2, 3, 4$, we have that

$$b_i = -7i^2 + 43i - 6.$$

3.1. **The coefficient** c**.** The coefficient c can be quickly found. Let C be a general curve of genus g and consider the curve $\overline{C} = \{[C, y] : y \in C\}$ in $\overline{\mathcal{M}}_{g,1}$ obtained varying the point y on C. Then the only generator class having non-zero intersection with $\overline{C}$ is ψ, and $\overline{C} \cdot \psi = 2g-2$. On the other hand, $\overline{C} \cdot \overline{\mathfrak{D}}^2_d$ is equal to the number of triples $(x, y, l) \in C \times C \times G^2_d(C)$ such that x and y are different points and $h^0(l(-x-y)) \geq 2$. The number of such linear series on a general C is computed by the Castelnuovo number (remember that $\rho = 0$), and for each of them the number of couples (x, y) imposing only one condition is twice the number of double points, computed by the Plücker formula. Hence we get the equation

$$\overline{\mathfrak{D}}^2_d \cdot \overline{C} = 2\left(\frac{(d-1)(d-2)}{2} - g\right) N_{g,2,d} = c\,(2g-2)$$

and so

$$c = \frac{2s(s-1)}{3s-1} N_{g,2,d}.$$

3.2. **The coefficients** a **and** b_0**.** In order to compute a and b_0, we use a Porteous-style argument. Let $\mathcal{G}^2_d$ be the family parametrizing triples (C, p, l), where $[C, p] \in \mathcal{M}^{\mathrm{irr}}_{g,1}$ and l is a $\mathfrak{g}^2_d$ on C; denote by $\eta : \mathcal{G}^2_d \to \mathcal{M}^{\mathrm{irr}}_{g,1}$ the natural map. There exists $\pi : \mathcal{Y}^2_d \to \mathcal{G}^2_d$ a universal pointed quasi-stable curve, with $\sigma : \mathcal{G}^2_d \to \mathcal{Y}^2_d$ the marked section. Let $\mathscr{L} \to \mathcal{Y}^2_d$ be the universal line bundle of relative degree d together with the trivialization $\sigma^*(\mathscr{L}) \cong \mathscr{O}_{\mathcal{G}^2_d}$, and $\mathscr{V} \subset \pi_*(\mathscr{L})$ be the sub-bundle which over each point $(C, p, l = (L, V))$ in $\mathcal{G}^2_d$ restricts to V. (See [Kho07, §2] for more details.)

Furthermore let us denote by $\mathcal{Z}^2_d$ the family parametrizing $((C, p), x_1, x_2, l)$, where $[C, p] \in \mathcal{M}^{\mathrm{irr}}_{g,1}$, $x_1, x_2 \in C$ and l is a $\mathfrak{g}^2_d$ on C, and let $\mu, \nu : \mathcal{Z}^2_d \to \mathcal{Y}^2_d$ be defined as the maps that send $((C, p), x_1, x_2, l)$ respectively to $((C, p), x_1, l)$ and $((C, p), x_2, l)$.

Now given a linear series $l = (L, V)$, the natural map

$$\varphi : V \to H^0(L|_{p+x})$$

globalizes to

$$\widetilde{\varphi} : \mathscr{V} \to \mu_* \left(\nu^* \mathscr{L} \otimes \mathscr{O}/\mathscr{I}_{\Gamma_\sigma + \Delta}\right) =: \mathscr{M}$$

as a map of vector bundle over $\mathcal{Y}^2_d$, where Δ and Γ_σ are the loci in $\mathcal{Z}^2_d$ determined respectively by $x_1 = x_2$ and $x_2 = p$. Then $\overline{\mathfrak{D}}^2_d \cap \mathcal{M}^{\mathrm{irr}}_{g,1}$ is the push-forward of the locus in $\mathcal{Y}^2_d$ where $\widetilde{\varphi}$ has rank ≤ 1. Using Porteous formula, we have

$$(1) \quad \left[\overline{\mathfrak{D}}^2_d\right]\Big|_{\mathcal{M}^{\mathrm{irr}}_{g,1}} = \eta_* \pi_* \left[\frac{\mathscr{V}^\vee}{\mathscr{M}^\vee}\right]_2 = \eta_* \pi_* \left(\pi^* c_2(\mathscr{V}^\vee) + \pi^* c_1(\mathscr{V}^\vee) \cdot c_1(\mathscr{M}) + c_1^2(\mathscr{M}) - c_2(\mathscr{M})\right).$$

Let us find the Chern classes of $\mathscr{M}$. Tensoring the exact sequence

$$0 \to \mathscr{I}_\Delta/\mathscr{I}_{\Delta+\Gamma_\sigma} \to \mathscr{O}/\mathscr{I}_{\Delta+\Gamma_\sigma} \to \mathscr{O}_\Delta \to 0$$

by $\nu^*\mathscr{L}$ and applying μ_*, we deduce that

$$\begin{aligned} ch(\mathscr{M}) &= ch(\mu_*(\mathscr{O}_{\Gamma_\sigma}(-\Delta) \otimes \nu^*\mathscr{L})) + ch(\mu_*(\mathscr{O}_\Delta \otimes \nu^*\mathscr{L})) \\ &= ch(\mu_*(\mathscr{O}_{\Gamma_\sigma}(-\Delta))) + ch(\mu_*(\mathscr{O}_\Delta \otimes \nu^*\mathscr{L})) \\ &= e^{-\sigma} + ch(\mathscr{L}) \end{aligned}$$

hence

$$\begin{aligned} c_1(\mathscr{M}) &= c_1(\mathscr{L}) - \sigma \\ c_2(\mathscr{M}) &= -\sigma c_1(\mathscr{L}). \end{aligned}$$

The following classes

$$\begin{aligned} \alpha &= \pi_* \left(c_1(\mathscr{L})^2 \cap \left[\mathcal{Y}^2_d\right]\right) \\ \gamma &= c_1(\mathscr{V}) \cap \left[\mathcal{G}^2_d\right] \end{aligned}$$

have been studied in [Kho07, Thm. 2.11]. In particular

$$\begin{aligned} \frac{6(g-1)(g-2)}{dN_{g,2,d}} \eta_*(\alpha)|_{\mathcal{M}^{\mathrm{irr}}_{g,1}} &= 6(gd - 2g^2 + 8d - 8g + 4)\lambda \\ &\quad + (2g^2 - gd + 3g - 4d - 2)\delta_0 \\ &\quad - 6d(g-2)\psi, \\ \frac{2(g-1)(g-2)}{N_{g,2,d}} \eta_*(\gamma)|_{\mathcal{M}^{\mathrm{irr}}_{g,1}} &= \left(-(g+3)\xi + 40\right)\lambda \\ &\quad + \frac{1}{6}\left((g+1)\xi - 24\right)\delta_0 \\ &\quad - 3d(g-2)\psi, \end{aligned}$$

where

$$\xi = 3(g-1) + \frac{(g+3)(3g-2d-1)}{g-d+5}.$$

Plugging into (1) and using the projection formula, we find

$$\begin{aligned} \left[\overline{\mathfrak{D}}^2_d\right]\Big|_{\mathcal{M}^{\mathrm{irr}}_{g,1}} &= \eta_* \left(-\gamma \cdot \pi_* c_1(\mathscr{L}) + \gamma \cdot \pi_*\sigma + \alpha + \pi_*\sigma^2 - \pi_*(\sigma c_1(\mathscr{L}))\right) \\ &= (1-d)\eta_*(\gamma) + \eta_*(\alpha) - N_{g,2,d} \cdot \psi. \end{aligned}$$

Hence

$$\begin{aligned} a &= \frac{48s^4 + 80s^3 - 16s^2 - 64s + 24}{(3s-1)(3s-2)(s+3)} N_{g,2,d} \\ b_0 &= \frac{24s^4 + 23s^3 - 18s^2 - 11s + 6}{3(3s-1)(3s-2)(s+3)} N_{g,2,d} \end{aligned}$$

and we recover the previously computed coefficient c.

3.3. **The coefficient** b_1**.** Let C be a general curve of genus $g-1$ and (E,p,q) a two-pointed elliptic curve, with $p-q$ not a torsion point in $\mathrm{Pic}^0(E)$. Let $\overline{C}_1 := \{(C \cup_{y\sim q} E, p)\}_{y\in C}$ be the family of curves obtained identifying the point $q \in E$ with a moving point $y \in C$. Computing the intersection of the divisor $\overline{\mathfrak{D}}^2_d$ with $\overline{C}_1$ is equivalent to answering the following question: how many triples (x,y,l) are there, with $y \in C$, $x \in C \cup_{y\sim q} E \setminus \{p\}$ and $l = \{l_C, l_E\}$ a limit $\mathfrak{g}^2_d$ on $C \cup_{y\sim q} E$, such that (p,x,l) arises as limit of (p_t, x_t, l_t) on a family of curves $\{C_t\}_t$ with smooth general element, where p_t and x_t impose only one condition on l_t a $\mathfrak{g}^2_d$?

Let $a^{l_E}(q) = (a_0, a_1, a_2)$ be the vanishing sequence of $l_E \in G^2_d(E)$ at q. Since C is general, there are no $\mathfrak{g}^2_{d-1}$ on C, hence l_C is base-point free and $a_2 = d$. Moreover we know $a_1 \leq d-2$. Let us suppose $x \in E \setminus \{q\}$. We distinguish two cases. If $\rho(E,q) = \rho(C,y) = 0$, then $w^{l_E}(q) = \rho(1,2,d) = 3d-8$. Thus $a^{l_E}(q) = (d-3, d-2, d)$. Removing the base point we have that $l_E(-(d-3)q)$ is a $\mathfrak{g}^2_3$ and $l_E(-(d-3)q-p-x)$ produces a $\mathfrak{g}^1_1$ on E, hence a contradiction. The other case is $\rho(E,q) = 1$ and $\rho(C,y) \leq -1$. These force $a^{l_E}(q) = (d-4, d-2, d)$ and $a^{l_C}(y) \geq (0,2,4)$. On E we have that $l_E(-(d-4)q-p-x)$ is a $\mathfrak{g}^1_2$.

The question splits in two: firstly, how many linear series $l_E \in G^2_4(E)$ and points $x \in E \setminus \{q\}$ are there such that $a^{l_E}(q) = (0,2,4)$ and $l_E(-p-x) \in G^1_2(E)$? The first condition restricts our attention to the linear series $l_E = (\mathscr{O}(4q), V)$ where V is a tridimensional vector space and $H^0(\mathscr{O}(4q-2q)) \subset V$, while the second condition tells us $H^0(\mathscr{O}(4q-p-x)) \subset V$. If $x = p$, then we get $p-q$ is a torsion point in $\mathrm{Pic}^0(E)$, a contradiction. On the other hand, if $x \in E \setminus \{p,q\}$, then $H^0(\mathscr{O}(4q-2q)) \cap H^0(\mathscr{O}(4q-p-x)) \neq \emptyset$ entails $p+x \equiv 2q$. Hence the point x and the space $V = H^0(\mathscr{O}(4q-2q)) + H^0(\mathscr{O}(4q-p-x))$ are uniquely determined.

Secondly, how many couples $(y, l_C) \in C \times G^2_d(C)$ are there, such that the vanishing sequence of l_C at y is greater than or equal to $(0,2,4)$? This is a particular case of a problem discussed in [Far09, Proof of Thm. 4.6]. The answer is

$$(g-1)\left(15N_{g-1,2,d,(0,2,2)} + 3N_{g-1,2,d,(1,1,2)} + 3N_{g-1,2,d,(0,1,3)}\right) = \frac{24(2s^2+3s-4)}{s+3} N_{g,2,d}.$$

Now let us suppose $x \in C \setminus \{y\}$. The condition on x and p can be reformulated in the following manner. We consider the curve $C \cup_y E$ as the special fiber X_0 of a family of curves $\pi : X \to B$ with sections $x(t)$ and $p(t)$ such that $x(0) = x$, $p(0) = p$, and with smooth general fiber having $l = (\mathscr{L}, V)$ a $\mathfrak{g}^2_d$ such that $l(-x-p)$ is a $\mathfrak{g}^1_{d-2}$. Let $V' \subset V$ be the two dimensional linear subspace formed by those sections $\sigma \in V$ such that $\mathrm{div}(\sigma) \geq x+p$. Then V' specializes on X_0 to $V'_C \subset V_C$ and $V'_E \subset V_E$ two-dimensional subspaces, where $\{l_C = (\mathscr{L}_C, V_C), l_E = (\mathscr{L}_E, V_E)\}$ is a limit $\mathfrak{g}^2_d$, such that

$$\begin{cases} \mathrm{ord}_y(\sigma_C) + \mathrm{ord}_y(\sigma_E) \geq d \\ \mathrm{div}(\sigma_C) \geq x \\ \mathrm{div}(\sigma_E) \geq p \end{cases}$$

for every $\sigma_C \in V'_C$ and $\sigma_E \in V'_E$. Let $l'_C := (\mathscr{L}_C, V'_C)$ and $l'_E := (\mathscr{L}_E, V'_E)$. Note that since $\sigma_E \geq p$, we get $\mathrm{ord}_y(\sigma_E) < d$, $\forall \sigma_E \in V'_E$. Then $\mathrm{ord}_y(\sigma_C) > 0$, hence $\mathrm{ord}_y(\sigma_C) \geq 2$, since y is a cuspidal point on C. Removing the base point, l'_C is a $\mathfrak{g}^1_{d-2}$ such that $l'_C(-x)$ is a $\mathfrak{g}^1_{d-3}$. Let us suppose $\rho(E,y) = 1$ and $\rho(C,y) = -1$. Then $a^{l_E}(y) = (d-4, d-2, d)$, $a^{l'_E}(y) = (d-4, d-2)$, $a^{l_C}(y) = (0,2,4)$ and $a^{l'_C}(y) = (2,4)$. Now l_C is characterized by the conditions $H^0(l_C(-2y-x)) \geq 2$ and $H^0(l_C(-4y-x)) \geq 1$. By Thm. 3 this possibility does not occur.

Suppose now $\rho(E,y) = \rho(C,y) = 0$. Then $a^{l_E}(y) = (d-3, d-2, d)$, i.e. $l_E(-(d-3)y) = |3y|$ is uniquely determined. On the C aspect we have that $a^{l_C}(y) = (0,2,3)$

and $h^0(l_C(-2y-x)) \geq 2$. Hence we are interested on Y, the locus of triples (x, y, l_C) such that the map

$$\varphi : H^0(l_C) \to H^0(l_C|_{2y+x})$$

has rank ≤ 1. By Thm. 2 there is only a finite number of such triples, and clearly the case $a^{l_C}(y) > (0,2,3)$ cannot occur. Moreover, note that x and y will be necessarily distinct.

Let $\mu = \pi_{1,2,4} : C \times C \times C \times W^2_d(C) \to C \times C \times W^2_d(C)$ and $\nu = \pi_{3,4} : C \times C \times C \times W^2_d(C) \to C \times W^2_d(C)$ be the natural projections respectively on the first, second and fourth components, and on the third and fourth components. Let $\pi : C \times C \times W^2_d(C) \to W^2_d(C)$ be the natural projection on the third component. Now φ globalizes to

$$\widetilde{\varphi} : \pi^*\mathscr{E} \to \mu_* \left(\nu^*\mathscr{L} \otimes \mathscr{O}/\mathscr{I}_{\mathcal{D}}\right) =: \mathscr{M}$$

as a map of rank 3 bundles over $C \times C \times W^2_d(C)$, where $\mathcal{D}$ is the pullback to $C \times C \times C \times W^2_d(C)$ of the divisor on $C \times C \times C$ that on $(x, y, C) \cong C$ restricts to $x + 2y$, $\mathscr{L}$ is a Poincaré bundle on $C \times W^2_d$ and $\mathscr{E}$ is the push-forward of $\mathscr{L}$ to $W^2_d(C)$. Then Y is the degeneracy locus where $\widetilde{\varphi}$ has rank ≤ 1. Let $\mathfrak{c}_i := c_i(\mathscr{E})$ be the Chern classes of $\mathscr{E}$. By Porteous formula, we have

$$[Y] = \begin{bmatrix} e_2 & e_3 \\ e_1 & e_2 \end{bmatrix}$$

where the e_i's are the Chern classes of $\pi^*\mathscr{E}^\vee - \mathscr{M}^\vee$, i.e.

$$\begin{aligned} e_1 &= \mathfrak{c}_1 + c_1(\mathscr{M}) \\ e_2 &= \mathfrak{c}_2 + \mathfrak{c}_1 c_1(\mathscr{M}) + c_1^2(\mathscr{M}) - c_2(\mathscr{M}) \\ e_3 &= \mathfrak{c}_3 + \mathfrak{c}_2 c_1(\mathscr{M}) + \mathfrak{c}_1 \left(c_1^2(\mathscr{M}) - c_2(\mathscr{M})\right) \\ &\quad + \left(c_1^3(\mathscr{M}) + c_3(\mathscr{M}) - 2c_1(\mathscr{M})c_2(\mathscr{M})\right). \end{aligned}$$

Let us find the Chern classes of $\mathscr{M}$. First we develop some notation (see also [ACGH85, §VIII.2]). Let $\pi_i : C \times C \times C \times W^2_d(C) \to C$ for $i = 1, 2, 3$ and $\pi_4 : C \times C \times C \times W^2_d(C) \to W^2_d(C)$ be the natural projections. Denote by θ the pull-back to $C \times C \times C \times W^2_d(C)$ of the class $\theta \in H^2(W^2_d(C))$ via π_4, and denote by η_i the cohomology class $\pi_i^*([\text{point}]) \in H^2(C \times C \times C \times W^2_d(C))$, for $i = 1, 2, 3$. Note that $\eta_i^2 = 0$. Furthermore, given a symplectic basis $\delta_1, \ldots, \delta_{2(g-1)}$ for $H^1(C, \mathbb{Z}) \cong H^1(W^2_d(C), \mathbb{Z})$, denote by δ^i_α the pull-back to $C \times C \times C \times W^2_d(C)$ of δ_α via π_i, for $i = 1, 2, 3, 4$. Let us define

$$\gamma_{ij} := -\sum_{\alpha=1}^{g-1} \left(\delta^j_\alpha \delta^i_{g-1+\alpha} - \delta^j_{g-1+\alpha}\delta^i_\alpha\right).$$

Note that

$$\begin{aligned} \gamma_{ij}^2 &= -2(g-1)\eta_i\eta_j \quad \text{and} \quad \eta_i\gamma_{ij} = \gamma_{ij}^3 = 0 \quad \text{for} \quad 1 \leq i < j \leq 3, \\ \gamma_{k4}^2 &= -2\eta_k\theta \quad \text{and} \quad \eta_k\gamma_{k4} = \gamma_{k4}^3 = 0 \quad \text{for} \quad k = 1, 2, 3. \end{aligned}$$

Moreover

$$\gamma_{ij}\gamma_{jk} = \eta_j\gamma_{ik},$$

for $1 \leq i < j < k \leq 4$. With this notation, we have

$$ch(\nu^*\mathscr{L} \otimes \mathscr{O}/\mathscr{I}_{\mathcal{D}}) = (1 + d\eta_3 + \gamma_{34} - \eta_3\theta)\left(1 - e^{-(\eta_1+\gamma_{13}+\eta_3+2\eta_2+2\gamma_{23}+2\eta_3)}\right),$$

hence by Grothendieck-Riemann-Roch

$$\begin{aligned} ch(\mathscr{M}) &= \mu_*\left((1 + (2-g)\eta_3)ch(\nu^*\mathscr{L} \otimes \mathscr{O}/\mathscr{I}_{\mathcal{D}})\right) \\ &= 3 + (d-2)\eta_1 + (2g + 2d - 6)\eta_2 - 2\gamma_{12} + \gamma_{14} + 2\gamma_{24} \\ &\quad - \eta_1\theta - 2\eta_2\theta + (8 - 2d - 4g)\eta_1\eta_2 - 2\eta_1\gamma_{24} - 2\eta_2\gamma_{14} + 2\eta_1\eta_2\theta. \end{aligned}$$

Using Newton's identities, we recover the Chern classes of $\mathscr{M}$:

$$\begin{aligned} c_1(\mathscr{M}) &= (d-2)\eta_1 + (2g+2d-6)\eta_2 - 2\gamma_{12} + \gamma_{14} + 2\gamma_{24}, \\ c_2(\mathscr{M}) &= (2d^2-8d+2gd+8-4g)\eta_1\eta_2 + (2g+2d-8)\eta_2\gamma_{14} \\ &\quad + (2d-4)\eta_1\gamma_{24} + 2\gamma_{14}\gamma_{24} - 2\eta_2\theta, \\ c_3(\mathscr{M}) &= (4-2d)\eta_1\eta_2\theta - 2\eta_2\gamma_{14}\theta. \end{aligned}$$

We finally find

$$\begin{aligned} [Y] &= \eta_1\eta_2(\mathfrak{c}_1^2(2d^2-8d+2dg+4-4(g-1)) \\ &\quad + \mathfrak{c}_1\theta(-12d-4g+40) + \mathfrak{c}_2(-4d+16-8g) + 12\theta^2) \\ &= \frac{(28s+48)(s-2)(s-1)}{(s+3)} N_{g,2,d} \cdot \eta_1\eta_2\theta^{g-1} \end{aligned}$$

where we have used the following identities proved in [Far09, Lemma 2.6]

$$\begin{aligned} \mathfrak{c}_1^2 &= \left(1 + \frac{2s+2}{s+3}\right)\mathfrak{c}_2 \\ \mathfrak{c}_1\theta &= (s+1)\mathfrak{c}_2 \\ \theta^2 &= \frac{(s+1)(s+2)}{3}\mathfrak{c}_2 \\ \mathfrak{c}_2 &= N_{g,2,d} \cdot \theta^{g-1}. \end{aligned}$$

We are going to show that we have already considered all non zero contributions. Indeed let us suppose $x = y$. Blowing up the point x, we obtain $C \cup_y \mathbb{P}^1 \cup_q E$ with $x \in \mathbb{P}^1 \setminus \{y, q\}$ and $p \in E \setminus \{q\}$. We reformulate the condition on x and p viewing our curve as the special fiber of a family of curves $\pi : X \to B$ as before. Let $\{l_C, l_{\mathbb{P}^1}, l_E\}$ be a limit $\mathfrak{g}^2_d$. Now V' specializes to V'_C, $V'_{\mathbb{P}^1}$ and V'_E. There are three possibilities: either $\rho(C, y) = \rho(\mathbb{P}^1, x, y, q) = \rho(E, p, q) = 0$, or $\rho(C, y) = -1$, $\rho(\mathbb{P}^1, x, y, q) = 0$, $\rho(E, p, q) = 1$, or $\rho(C, y) = -1$, $\rho(\mathbb{P}^1, x, y, q) = 1$, $\rho(E, p, q) = 0$. In all these cases $a^{l_C}(y) = (0, 2, a_2^{l_C}(y))$ (remember that l_C is base-point free) and $a^{l_E}(q) = (a_0^{l_E}(q), d-2, d)$. Hence $a^{l_{\mathbb{P}^1}}(y) = (a_0^{l_{\mathbb{P}^1}}(y), d-2, d)$ and $a^{l_{\mathbb{P}^1}}(q) = (0, 2, a_2^{l_{\mathbb{P}^1}}(q))$. Let us restrict now to the sections in V'_C, $V'_{\mathbb{P}^1}$ and V'_E. For all sections $\sigma_{\mathbb{P}^1} \in V'_{\mathbb{P}^1}$ since $\mathrm{div}(\sigma_{\mathbb{P}^1}) \geq x$, we have that $\mathrm{ord}_y(\sigma_{\mathbb{P}^1}) < d$ and hence $\mathrm{ord}_y(\sigma_{\mathbb{P}^1}) \leq d-2$. On the other side, since for all $\sigma_E \in V'_E$, $\mathrm{div}(\sigma_E) \geq p$, we have that $\mathrm{ord}_q(\sigma_E) < d$ and hence $\mathrm{ord}_q(\sigma_{\mathbb{P}^1}) \geq 2$. Let us take one section $\tau \in V'_{\mathbb{P}^1}$ such that $\mathrm{ord}_y(\tau) = d-2$. Since $\mathrm{div}(\tau) \geq (d-2)y + x$, we get $\mathrm{ord}_q(\tau) \leq 1$, hence a contradiction.

Thus we have that

$$\overline{\mathfrak{D}}^2_d \cdot \overline{C}_1 = \frac{24(2s^2+3s-4)}{s+3} N_{g,2,d} + \frac{(28s+48)(s-2)(s-1)}{(s+3)} N_{g,2,d}.$$

while considering the intersection of the test curve $\overline{C}_1$ with the generating classes we have

$$\overline{\mathfrak{D}}^2_d \cdot \overline{C}_1 = b_1(2g-4),$$

whence

$$b_1 = \frac{14s^3+6s^2-8s}{(3s-2)(s+3)} N_{g,2,d}.$$

Remark 5. The previous class $[Y]$ being nonzero, it implies together with Thm. 2 that the scheme $\mathcal{G}^2_d((0,2,3))$ over $\mathcal{M}_{g-1,1}$ is not irreducible.

3.4. **The coefficient b_{g-1}.** We analyze now the following test curve $\overline{E}$. Let (C,p) be a general pointed curve of genus $g-1$ and (E,q) be a pointed elliptic curve. Let us identify the points p and q and let y be a movable point in E. We have

$$0 = \overline{\mathfrak{D}}^2_d \cdot \overline{E} = c + b_1 - b_{g-1},$$

whence

$$b_{g-1} = \frac{48s^4 + 12s^3 - 56s^2 + 20s}{(3s-1)(3s-2)(s+3)} N_{g,2,d}.$$

3.5. **A test.** Furthermore, as a test we consider the family of curves R. Let (C,p,q) be a general two-pointed curve of genus $g-1$ and let us identify the point q with the base point of a general pencil of plane cubic curves. We have

$$0 = \overline{\mathfrak{D}}^2_d \cdot R = a - 12b_0 + b_{g-1}.$$

3.6. **The remaining coefficients in case $g=6$.** Denote by P_g the moduli space of stable g-pointed rational curves. Let (E,p,q) be a general two-pointed elliptic curve and let $j : P_g \to \overline{\mathcal{M}}_{g,1}$ be the map obtained identifying the first marked point on a rational curve with the point $q \in E$ and attaching a fixed elliptic tail at the other marked points. We claim that $j^*(\overline{\mathfrak{D}}^2_6) = 0$.

Indeed consider a flag curve of genus 6 in the image of j. Clearly the only possibility for the adjusted Brill-Noether numbers is to be zero on each aspect. In particular the collection of the aspects on all components but E smooths to a $\mathfrak{g}^2_6$ on a general one-pointed curve of genus 5. As discussed in section 3.3, the point x can not be in E. Suppose x is in the rest of the curve. Then smoothing we get l a $\mathfrak{g}^2_6$ on a general pointed curve of genus 5 such that $l(-2q-x))$ is a $\mathfrak{g}^1_3$, a contradiction.

Now let us study the pull-back of the generating classes. As in [EH87, §3] we have that $j^*(\lambda) = j^*(\delta_0) = 0$. Furthermore $j^*(\psi) = 0$.

For $i = 1,\ldots,g-3$ denote by $\varepsilon^{(1)}_i$ the class of the divisor which is the closure in P_g of the locus of two-component curves having exactly the first marked point and other i marked points on one of the two components. Then clearly $j^*(\delta_i) = \varepsilon^{(1)}_{i-1}$ for $i = 2,\ldots,g-2$. Moreover adapting the argument in [EH89, pg. 49], we have that

$$j^*(\delta_{g-1}) = -\sum_{i=1}^{g-3} \frac{i(g-i-1)}{g-2} \varepsilon^{(1)}_i$$

while

$$j^*(\delta_1) = -\sum_{i=1}^{g-3} \frac{(g-i-1)(g-i-2)}{(g-1)(g-2)} \varepsilon^{(1)}_i.$$

Finally since $j^*(\overline{\mathfrak{D}}^2_6) = 0$, checking the coefficient of $\varepsilon^{(1)}_i$ we obtain

$$b_{i+1} = \frac{(g-i-1)(g-i-2)}{(g-1)(g-2)} b_1 + \frac{i(g-i-1)}{g-2} b_{g-1}$$

for $i = 1,2,3$.

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