Graph Theory Solutions by Richard

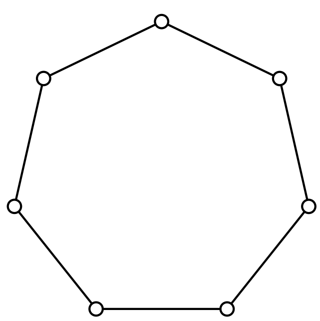
1.1.3 Order the vertices of $K_{m,n}$ as $x_1, x_2, x_3, \ldots, x_m$, $y_1, y_2, y_3, \ldots, y_n$, where the vertices x_i are in the partite set of size m, and the vertices y_i are in the partite set of size n. Relative to this ordering, the adjacency matrix of $K_{m,n}$ has the following form.

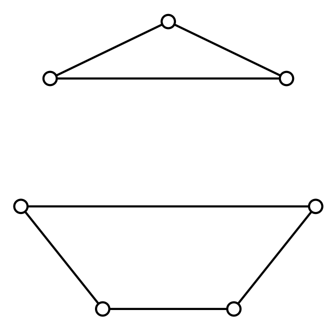
$$\left[\begin{array}{ccccc|ccccc}
0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 \\
\hline
1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\
1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\
1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0
\end{array}\right]$$

The upper-left block of zeros is $m \times m$. The upper-right block of ones is $m \times n$. The lower-left block of ones is $n \times m$. The lower-right block of zeros is $n \times n$.

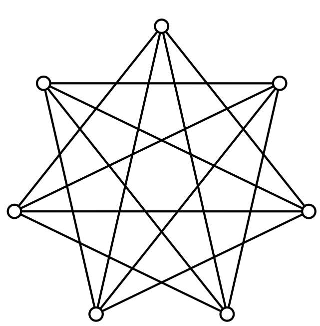
1.1.17 Determine the number of isomorphism classes of 7-vertex graphs for which every vertex has degree 4.

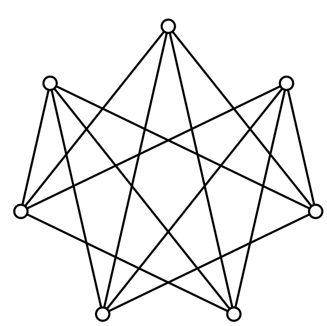
It is easier to think about the complements of such graphs. They are the 7-vertex graphs for which every vertex has degree 2. There are just two of them, illustrated below.





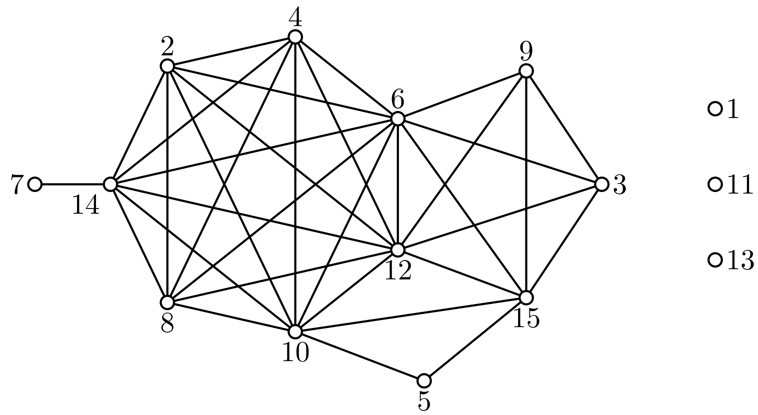
Now, taking complements, we get just two isomorphism classes of 7-vertex graphs for which every vertex has degree 4.





1.2.3 In this problem $V(G) = \{1, 2, 3, 4, \ldots, 15\}$. Two vertices are adjacent if and only if they have a common factor greater than 1.

Here is a picture of G:



Number of components: **Four.** (Note the three isolated vertices.)

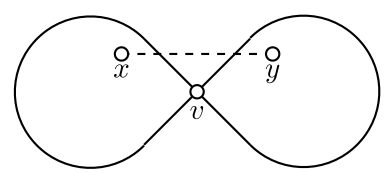
Length of longest path: The path $7, 14, 2, 8, 4, 6, 12, 10, 5, 15, 3, 9$ has **length 11.**

1.2.20 If v is a cut vertex of G, then $\overline{G} - v$ is connected.

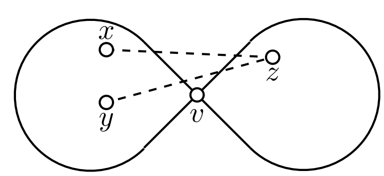
Proof. Suppose v is a cut vertex of G. We need to show that $\overline{G} - v$ is connected. To do this we must show that given any two vertices x and y of $\overline{G} - v$, there is a path in $\overline{G} - v$ that joins x to y.

Thus let x and y be two vertices of $\overline{G} - v$. Note that, by definition of complement, x and y are also two vertices of $G - v$. Now, $G - v$ is disconnected, since v is a cut vertex of G. We consider two mutually exclusive and exhaustive cases.

CASE A. Vertices x and y are in different components of $G - v$. In this case the edge xy cannot be an edge of G. Thus xy is an edge of $\overline{G}$. But neither x nor y equals v, so xy is an edge of $\overline{G} - v$. Thus we have produced a path xy (of length 1) in $\overline{G} - v$, joining x to y.



CASE B. Vertices x and y are in the same component of $G - v$. Let z be a vertex in a component different from the component that x and y are in. Reasoning as in the first case, it follows that xz and zy are edges of $\overline{G} - v$. Thus we have produced a path x, z, y (of length 2) in $\overline{G} - v$, joining x to y.



The above cases show that for any vertices x and y of $\overline{G} - v$, there is a path in $\overline{G} - v$ joining x to y. Therefore $\overline{G} - v$ is connected. ■

1.2.22 A graph G is connected if and only if for every partition of its vertices into two non-empty subsets, there is an edge with an endpoint in each set.

Proof.

($\Longrightarrow$) Suppose G is connected. Take an arbitrary partition $V(G) = X \cup Y$ of $V(G)$ into two non-empty sets. We need to show that G has an edge with one endpoint in X and the other in Y. Select vertices x in X, and y in Y. (Possible because X and Y are non-empty.) Since G is connected, there has to be a path in G that joins x to y. Denote this path as follows.

$$x = x_0, x_1, x_2, x_3, \ldots, x_n = y$$

The first vertex x_0 of this path is in X, and the last vertex x_n is in Y. Any one of the others is either in X or Y. Let i be the smallest index for which $x_i \in Y$. (Such an i exists, because $x_n \in Y$, so i is at most n.) Now we have $x_{i-1} \in X$ and $x_i \in Y$, so $x_{i-1}x_i$ is an edge of G with one endpoint in X and the other in Y.

($\Longleftarrow$) Suppose that for any partition $V(G) = X \cup Y$ of $V(G)$ into two non-empty sets, G has an edge with one endpoint in X and the other in Y. We need to show G is connected. For the sake of contradiction, suppose G is not connected. Let C be one of its components. Now we have a partition

$$V(G) = V(C) \cup (V(G) - V(C))$$

of $V(G)$ into two non-empty sets. By assumption, G has an edge with endpoints in each set in this partition. That is to say G has an edge with one endpoint in one of its components and the other endpoint in another component. This is a contradiction. ■