1 The Linearized Einstein Equations

1.1 The Assumption

1.1.1 Simplest Version

The simplest version of the linearized theory begins with flat Minkowski spacetime with basis vectors

$$\partial_\mu = \frac{\partial}{\partial x^\mu}$$

and metric tensor components

$$\eta_{\mu\nu} = \begin{cases} -1 & \text{for} \quad \mu = \nu = 0 \\ 0 & \text{for} \quad \mu \neq \nu \\ 1 & \text{for} \quad \mu = \nu = 1, 2, 3 \end{cases}$$

and assumes that the actual spacetime metric has the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

where the $h_{\mu\nu}$ are all much less than 1. One then makes this substitution everywhere in the theory and discards any terms that contain more than one factor of $h_{\mu\nu}$. For example, the exact expression for the metric compatible connection coefficients in the holonomic frame

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\sigma} \left(\partial_\gamma g_{\sigma\beta} + \partial_\beta g_{\gamma\sigma} - \partial_\sigma g_{\beta\gamma} \right)$$

becomes just

$$\Gamma_{\text{lin}}{}^\alpha{}_{\beta\gamma} = \frac{1}{2} \eta^{\alpha\sigma} \left(\partial_\gamma h_{\sigma\beta} + \partial_\beta h_{\gamma\sigma} - \partial_\sigma h_{\beta\gamma} \right)$$

and the exact expression for the curvature tensor components

$$R^\alpha{}_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha{}_{\beta\nu} - \partial_\nu \Gamma^\alpha{}_{\beta\mu} + \Gamma^\alpha{}_{\sigma\mu} \Gamma^\sigma{}_{\beta\nu} - \Gamma^\alpha{}_{\sigma\nu} \Gamma^\sigma{}_{\beta\mu}$$

loses the bothersome products of connection coefficients and becomes just

$$R_{\text{lin}}{}^\alpha{}_{\beta\mu\nu} = \partial_\mu \Gamma_{\text{lin}}{}^\alpha{}_{\beta\nu} - \partial_\nu \Gamma_{\text{lin}}{}^\alpha{}_{\beta\mu}.$$

1.1.2 Fancier Versions

Instead of simply counting factors of h and throwing away terms with too many, we can be more systematic and imagine a one-parameter family of metric tensors $g_{\mu\nu}(\varepsilon)$ with

$$\begin{aligned} g_{\mu\nu}(0) &= \eta_{\mu\nu} \\ g_{\mu\nu}(\varepsilon) &= \eta_{\mu\nu} + \varepsilon h_{\mu\nu} \end{aligned}$$

and then expand everything as a Taylor Series in the parameter ε. That is just the same as taking variational derivatives with

$$\delta g_{\mu\nu} = h_{\mu\nu}$$

and

$$
\begin{aligned}
\Gamma_{\text{lin}}{}^{\alpha}{}_{\beta\gamma} &= \delta\Gamma^{\alpha}{}_{\beta\gamma} \\
R_{\text{lin}}{}^{\alpha}{}_{\beta\mu\nu} &= \delta R^{\alpha}{}_{\beta\mu\nu}
\end{aligned}
$$

1.2 Coordinate or Gauge Variations

1.2.1 From Coordinate Transformations

Suppose that the coordinates are varied at the same time that the metric tensor is varied. Thus, the coordinates become

$$x^{\mu'} = x^{\mu} + \varepsilon\xi^{\mu}$$

$$g_{\alpha'\beta'} = \frac{\partial x^{\mu}}{\partial x^{\alpha'}} g_{\mu\nu} \frac{\partial x^{\nu}}{\partial x^{\beta'}}$$

Solve the coordinate relation and take the partial derivatives.

$$x^{\mu} = x^{\mu'} - \varepsilon\xi^{\mu}$$

$$\frac{\partial x^{\mu}}{\partial x^{\alpha'}} = \delta^{\mu'}_{\alpha'} - \varepsilon\partial_{\alpha'}\xi^{\mu}$$

$$\frac{\partial x^{\nu}}{\partial x^{\beta'}} = \delta^{\nu'}_{\beta'} - \varepsilon\partial_{\beta'}\xi^{\nu}$$

Weird notation alert! Notice that the primes are attached to the components and not to the indexes

$$
\begin{aligned}
g_{\alpha'\beta'} &= \left(\delta^{\mu'}_{\alpha'} - \varepsilon\partial_{\alpha'}\xi^{\mu}\right) g_{\mu\nu} \left(\delta^{\nu'}_{\beta'} - \varepsilon\partial_{\beta'}\xi^{\nu}\right) \\
&= \delta^{\mu'}_{\alpha'} g_{\mu\nu} \delta^{\nu'}_{\beta'} - \varepsilon\partial_{\alpha'}\xi^{\mu} g_{\mu\nu} \delta^{\nu'}_{\beta'} - \delta^{\mu'}_{\alpha'} g_{\mu\nu} \varepsilon\partial_{\beta'}\xi^{\nu} + o\left(\varepsilon^2\right) \\
&= g_{\alpha\beta} - \varepsilon\partial_{\alpha}\xi^{\mu} g_{\mu\beta} - \varepsilon g_{\alpha\nu}\partial_{\beta}\xi^{\nu} + o\left(\varepsilon^2\right) \\
&= g_{\alpha\beta} - \varepsilon\left(\partial_{\alpha}\xi_{\beta} + \partial_{\beta}\xi_{\alpha}\right)
\end{aligned}
$$

Finally, putting in the metric variation

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \varepsilon h_{\alpha\beta}$$

as well, we get

$$
\begin{aligned}
g_{\alpha'\beta'} &= \eta_{\alpha\beta} + \varepsilon h_{\alpha\beta} - \varepsilon\left(\partial_{\alpha}\xi_{\beta} + \partial_{\beta}\xi_{\alpha}\right) \\
&= \eta_{\alpha\beta} + \varepsilon\left(h_{\alpha\beta} - \partial_{\alpha}\xi_{\beta} - \partial_{\beta}\xi_{\alpha}\right)
\end{aligned}
$$

Thus, the metric variation

$$\delta g_{\alpha\beta} = h_{\alpha\beta} - \partial_{\alpha}\xi_{\beta} - \partial_{\beta}\xi_{\alpha}$$

differs from the variation

$$\delta g_{\alpha\beta} = h_{\alpha\beta}$$

by coordinate variations and therefore represents the same spacetime geometry. A change of the form

$$h_{\alpha\beta} \rightarrow h_{\alpha\beta} - \partial_\alpha \xi_\beta - \partial_\beta \xi_\alpha$$

is called a *gauge transformation* because it should have no effect on the physics.

Note that this transformation is a special case of something we have seen before: The Lie derivative of the metric tensor

$$\pounds_\xi g_{\alpha\beta} = -\xi_{\beta;\alpha} - \xi_{\alpha;\beta}$$

which expresses the result of dragging the metric along the integrals curves of the vector field ξ. For a general variation of the metric, a change of the form

$$\delta g_{\alpha\beta} \rightarrow \delta g_{\alpha\beta} + \pounds_\xi g_{\alpha\beta}$$

is purely a coordinate or gauge transformation.

1.2.2 Gauge Conditions

It is useful to exploit the coordinate arbitrariness to simplify the metric variation. Define the trace reversal operation

$$\bar{M}_{\alpha\beta} = M_{\alpha\beta} - \frac{1}{2}\left(\eta^{\rho\sigma} M_{\rho\sigma}\right) \eta_{\alpha\beta}$$

and notice that it has the properties

$$\eta^{\rho\sigma} \bar{M}_{\rho\sigma} = -\eta^{\rho\sigma} M_{\rho\sigma}$$

and

$$\bar{\bar{M}}_{\alpha\beta} = M_{\alpha\beta}$$

Now consider the conditions

$$\partial_\beta \bar{h}_\alpha{}^\beta = 0.$$

If these conditions are not satisfied by $\bar{h}_\alpha{}^\beta$ perform a coordinate transformation to

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - \left(\partial_\nu \xi_\mu + \partial_\mu \xi_\nu - \eta_{\mu\nu} \partial_\rho \xi^\rho\right)$$

or, with one index raised,

$$\bar{h}'^{\ \beta}_{\alpha} = \bar{h}_\alpha{}^\beta - \eta^{\beta\gamma}\left(\partial_\gamma \xi_\alpha + \partial_\alpha \xi_\gamma - \eta_{\alpha\gamma} \partial_\rho \xi^\rho\right)$$

The conditions are then

$$\partial_\beta \bar{h}_\alpha{}^\beta - \eta^{\beta\gamma} \partial_\beta\left(\partial_\gamma \xi_\alpha + \partial_\alpha \xi_\gamma - \eta_{\alpha\gamma} \partial_\rho \xi^\rho\right) = 0$$

or

$$\begin{aligned}
\eta^{\beta\gamma} \partial_\beta\left(\partial_\gamma \xi_\alpha + \partial_\alpha \xi_\gamma - \eta_{\alpha\gamma} \partial_\rho \xi^\rho\right) &= \partial_\beta \bar{h}_\alpha{}^\beta \\
\eta^{\beta\gamma} \partial_\beta \partial_\gamma \xi_\alpha + \eta^{\beta\gamma} \partial_\beta \partial_\alpha \xi_\gamma - \eta^{\beta\gamma} \partial_\beta \eta_{\alpha\gamma} \partial_\rho \xi^\rho &= \partial_\beta \bar{h}_\alpha{}^\beta \\
\eta^{\beta\gamma} \partial_\beta \partial_\gamma \xi_\alpha + \eta^{\beta\gamma} \partial_\beta \partial_\alpha \xi_\gamma - \delta_\alpha^\beta \partial_\beta \partial_\rho \xi^\rho &= \partial_\beta \bar{h}_\alpha{}^\beta \\
\eta^{\beta\gamma} \partial_\beta \partial_\gamma \xi_\alpha + \partial_\beta \partial_\alpha \xi^\beta - \delta_\alpha^\beta \partial_\beta \partial_\rho \xi^\rho &= \partial_\beta \bar{h}_\alpha{}^\beta \\
\eta^{\beta\gamma} \partial_\beta \partial_\gamma \xi_\alpha + \partial_\alpha \partial_\beta \xi^\beta - \partial_\alpha \partial_\rho \xi^\rho &= \partial_\beta \bar{h}_\alpha{}^\beta \\
\eta^{\beta\gamma} \partial_\beta \partial_\gamma \xi_\alpha &= \partial_\beta \bar{h}_\alpha{}^\beta
\end{aligned}$$

Define the d'Alembertian operator

$$\Box = \eta^{\beta\gamma}\partial_\beta\partial_\gamma$$

and get a set of wave equations for the coordinate fluctuations that are needed to satisfy these conditions:

$$\Box\xi_\alpha = \partial_\beta \bar{h}_\alpha{}^\beta$$

Thus, the conditions

$$\partial_\beta \bar{h}_\alpha{}^\beta = 0$$

can always be satisfied by an appropriate choice of coordinate variations.

Notice that the coordinate variations ξ_α that are needed to satisfy the conditions are determined only up to variations that satisfy

$$\Box\xi_\alpha = 0$$

Thus coordinate variations of this form preserve the gauge conditions.

1.3 Working Out the Linearized Curvature Components

1.3.1 The Full Riemann Tensor

Start with

$$R_{\mathrm{lin}}{}^\alpha{}_{\beta\mu\nu} = \partial_\mu \Gamma_{\mathrm{lin}}{}^\alpha{}_{\beta\nu} - \partial_\nu \Gamma_{\mathrm{lin}}{}^\alpha{}_{\beta\mu}$$

or equivalently

$$\delta R^\alpha{}_{\beta\mu\nu} = \partial_\mu \delta\Gamma^\alpha{}_{\beta\nu} - \partial_\nu \delta\Gamma^\alpha{}_{\beta\mu}$$

where

$$\delta\Gamma^\alpha{}_{\beta\nu} = \Gamma_{\mathrm{lin}}{}^\alpha{}_{\beta\nu} = \frac{1}{2}\eta^{\alpha\sigma}\left(\partial_\nu h_{\sigma\beta} + \partial_\beta h_{\nu\sigma} - \partial_\sigma h_{\beta\nu}\right)$$

and

$$\delta\Gamma^\alpha{}_{\beta\mu} = \frac{1}{2}\eta^{\alpha\sigma}\left(\partial_\mu h_{\sigma\beta} + \partial_\beta h_{\mu\sigma} - \partial_\sigma h_{\beta\mu}\right)$$

so that

$$\begin{aligned}
\delta R^\alpha{}_{\beta\mu\nu} &= \frac{1}{2}\eta^{\alpha\sigma}\left(\partial_\mu\left(\partial_\nu h_{\sigma\beta} + \partial_\beta h_{\nu\sigma} - \partial_\sigma h_{\beta\nu}\right) - \partial_\nu\left(\partial_\mu h_{\sigma\beta} + \partial_\beta h_{\mu\sigma} - \partial_\sigma h_{\beta\mu}\right)\right)\\
&= \frac{1}{2}\eta^{\alpha\sigma}\left(\partial_\mu\partial_\nu h_{\sigma\beta} + \partial_\mu\partial_\beta h_{\nu\sigma} - \partial_\mu\partial_\sigma h_{\beta\nu} - \partial_\nu\partial_\mu h_{\sigma\beta} - \partial_\nu\partial_\beta h_{\mu\sigma} + \partial_\nu\partial_\sigma h_{\beta\mu}\right)\\
&= \frac{1}{2}\eta^{\alpha\sigma}\left(\partial_\mu\partial_\beta h_{\nu\sigma} - \partial_\mu\partial_\sigma h_{\beta\nu} - \partial_\nu\partial_\beta h_{\mu\sigma} + \partial_\nu\partial_\sigma h_{\beta\mu}\right)
\end{aligned}$$

or

$$\delta R_{\sigma\beta\mu\nu} = \frac{1}{2}\left(\partial_\mu\partial_\beta h_{\nu\sigma} - \partial_\mu\partial_\sigma h_{\beta\nu} - \partial_\nu\partial_\beta h_{\mu\sigma} + \partial_\nu\partial_\sigma h_{\beta\mu}\right)$$

1.3.2 Contracted Curvature Tensors

Now contract to form the variation of the Ricci tensor

$$\begin{aligned}\delta R_{\beta\nu} &= \delta\left(g^{\sigma\mu}R_{\sigma\beta\mu\nu}\right)\\ &= \left(\delta g^{\sigma\mu}\right)R_{\sigma\beta\mu\nu} + g^{\sigma\mu}\delta R_{\sigma\beta\mu\nu}\end{aligned}$$

Since we are varying around flat Minkowski spacetime, the first term vanishes and we get

$$\begin{aligned}\delta R_{\beta\nu} &= \eta^{\sigma\mu}\frac{1}{2}\left(\partial_\mu\partial_\beta h_{\nu\sigma} - \partial_\mu\partial_\sigma h_{\beta\nu} - \partial_\nu\partial_\beta h_{\mu\sigma} + \partial_\nu\partial_\sigma h_{\beta\mu}\right)\\ &= \frac{1}{2}\left(\partial_\beta\left(\eta^{\sigma\mu}\partial_\mu h_{\nu\sigma}\right) - \eta^{\sigma\mu}\partial_\mu\partial_\sigma h_{\beta\nu} - \partial_\beta\partial_\nu\left(\eta^{\sigma\mu}h_{\mu\sigma}\right) + \partial_\nu\left(\eta^{\sigma\mu}\partial_\sigma h_{\beta\mu}\right)\right)\end{aligned}$$

Rearrange the terms.

$$2\delta R_{\beta\nu} = \partial_\beta\left(\eta^{\sigma\mu}\partial_\mu h_{\nu\sigma}\right) + \partial_\nu\left(\eta^{\sigma\mu}\partial_\sigma h_{\beta\mu}\right) - \partial_\beta\partial_\nu\left(\eta^{\sigma\mu}h_{\mu\sigma}\right) - \eta^{\sigma\mu}\partial_\mu\partial_\sigma h_{\beta\nu}$$

Rename the dummy indexes to make terms look alike.

$$2\delta R_{\beta\nu} = \partial_\beta\left(\eta^{\sigma\rho}\partial_\rho h_{\nu\sigma}\right) + \partial_\nu\left(\eta^{\rho\sigma}\partial_\rho h_{\beta\sigma}\right) - \partial_\beta\partial_\nu\left(\eta^{\sigma\mu}h_{\mu\sigma}\right) - \eta^{\sigma\mu}\partial_\mu\partial_\sigma h_{\beta\nu}$$

Define

$$h = \eta^{\sigma\mu}h_{\mu\sigma}$$

and

$$\eta^{\sigma\mu}\partial_\mu\partial_\sigma = \Box$$

so the expression becomes

$$2\delta R_{\beta\nu} = \partial_\beta\left(\eta^{\sigma\rho}\partial_\rho h_{\nu\sigma}\right) + \partial_\nu\left(\eta^{\rho\sigma}\partial_\rho h_{\beta\sigma}\right) - \partial_\beta\partial_\nu h - \Box h_{\beta\nu}$$

Split the trace term into two terms.

$$2\delta R_{\beta\nu} = \partial_\beta\left(\eta^{\sigma\rho}\partial_\rho h_{\nu\sigma}\right) - \frac{1}{2}\partial_\beta\partial_\nu h + \partial_\nu\left(\eta^{\rho\sigma}\partial_\rho h_{\beta\sigma}\right) - \frac{1}{2}\partial_\beta\partial_\nu h - \Box h_{\beta\nu}$$

and then notice that partial derivatives are common between adjacent terms so that

$$2\delta R_{\beta\nu} = \partial_\beta\left(\eta^{\sigma\rho}\partial_\rho h_{\nu\sigma} - \frac{1}{2}\partial_\nu h\right) + \partial_\nu\left(\eta^{\rho\sigma}\partial_\rho h_{\beta\sigma} - \frac{1}{2}\partial_\beta h\right) - \Box h_{\beta\nu}$$

Now notice that

$$\partial_\rho \bar{h}_\nu{}^\rho = \partial_\rho\left(h_\nu{}^\rho - \frac{1}{2}h\delta^\rho_\nu\right) = \eta^{\sigma\rho}\partial_\rho h_{\nu\sigma} - \frac{1}{2}\partial_\nu h$$

and

$$\partial_\rho \bar{h}_\beta{}^\rho = \partial_\rho\left(h_\beta{}^\rho - \frac{1}{2}h\delta^\rho_\beta\right) = \eta^{\sigma\rho}\partial_\rho h_{\beta\sigma} - \frac{1}{2}\partial_\beta h$$

so that the variation of the Ricci tensor becomes

$$2\delta R_{\beta\nu} = \partial_\beta \partial_\rho \bar{h}_\nu{}^\rho + \partial_\nu \partial_\rho \bar{h}_\beta{}^\rho - \Box h_{\beta\nu}$$

If we choose the gauge conditions

$$\partial_\rho \bar{h}_\nu{}^\rho = 0$$

then the result for the Ricci tensor variation is just

$$\delta R_{\beta\nu} = -\frac{1}{2}\Box h_{\beta\nu}$$

Since the Einstein tensor is just the trace reversed Ricci tensor

$$G_{\beta\nu} = \bar{R}_{\beta\nu}$$

we can get the variation of the Einstein tensor with no more work:

$$\delta G_{\beta\nu} = -\frac{1}{2}\Box \bar{h}_{\beta\nu}$$

1.4 Linearized Einstein Equations

The linearized version of the Einstein equations

$$G^{\mu\nu} = 8\pi k T^{\mu\nu}$$

is just

$$\delta G^{\mu\nu} = 8\pi k T^{\mu\nu}$$

or

$$-\Box \bar{h}^{\mu\nu} = 16\pi k T^{\mu\nu}$$

in the gauge specified by

$$\partial_\nu \bar{h}^{\mu\nu} = 0.$$

Written out, the linearized equations are

$$-\frac{\partial^2}{\partial t^2}\bar{h}^{\mu\nu} + \frac{\partial^2}{\partial x^2}\bar{h}^{\mu\nu} + \frac{\partial^2}{\partial y^2}\bar{h}^{\mu\nu} + \frac{\partial^2}{\partial z^2}\bar{h}^{\mu\nu} = 16\pi k T^{\mu\nu}(t, x, y, z)$$

so it is very clear that we have a wave equation for each of the components.

2 Waves

2.1 Plane Wave Expansions

2.1.1 Transverse Waves

So long as these are waves in a flat background spacetime, it makes sense to expand them in plane waves. A single plane wave would be

$$\bar{h}^{\mu\nu}(k, x) = \mathrm{Re}\left\{\bar{a}^{\mu\nu} e^{ik_\rho x^\rho}\right\}$$

where the $\bar{a}^{\mu\nu}$ are complex coefficients that contain the phase information as well as the amplitude information. For such a wave, the gauge condition is

$$\partial_\nu \bar{h}^{\mu\nu} = \mathrm{Re}\left\{ i\bar{a}^{\mu\nu} k_\nu e^{ik_\rho x^\rho} \right\} = 0$$

so that we need the "transverse wave" condition

$$\bar{a}^{\mu\nu} k_\nu = 0$$

and the d'Alembertian operator yields

$$\begin{aligned} \Box \bar{h}^{\mu\nu}(k,x) &= -\eta^{\alpha\beta} k_\alpha k_\beta \bar{h}^{\mu\nu}(k,x) \\ &= -(k\cdot k)\,\bar{h}^{\mu\nu}(k,x). \end{aligned}$$

2.1.2 Remaining Coordinate Freedom

Where there is no stress-energy, the linearized system is solved by a single plane wave with

$$k \cdot k = 0$$

Some coordinate variations are still allowed.

$$\bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} - \partial_\nu \xi_\mu - \partial_\mu \xi_\nu + \eta_{\mu\nu} \partial_\rho \xi^\rho$$

so long as

$$\Box \xi_\mu = 0$$

These variations preserve both the wave equation and the gauge condition. For a single plane wave, take

$$\xi_\mu = \mathrm{Re}\left\{ \ell_\mu e^{ik_\rho x^\rho} \right\}$$

with

$$k \cdot k = 0$$

so that

$$\bar{a}^{\mu\nu} \to \bar{a}^{\mu\nu} - ik^\mu \ell^\nu - ik^\nu \ell^\mu + i\eta^{\mu\nu} k_\rho \ell^\rho$$

The gauge condition is preserved since

$$\begin{aligned} \bar{a}^{\mu\nu} k_\nu &\to \bar{a}^{\mu\nu} k_\nu - ik^\mu \ell^\nu k_\nu - ik^\nu k_\nu \ell^\mu + i\eta^{\mu\nu} k_\nu k_\rho \ell^\rho \\ &= -ik^\mu \ell^\nu k_\nu + i\eta^{\mu\nu} k_\nu k_\rho \ell^\rho \\ &= -ik^\mu \ell^\nu k_\nu + ik^\mu \ell^\nu k_\nu = 0 \end{aligned}$$

2.1.3 Radiation Gauge

One way to fix the remaining freedom is to require

$$\bar{a}^{\mu 0} = 0$$

Although it looks as if there are four conditions here, remember that we already have one linear combination satisfied, namely

$$k_\mu \bar{a}^{\mu 0} = 0$$

To see what happens in detail, switch to a set of basis vectors that are aligned with the wave so that ∂_3 is in the propagation direction. In that case

$$k_1 = k_2 = 0$$

and

$$k_3 = \pm k_0$$

The transverse wave condition then reduces to

$$\bar{a}^{\mu 3} = \bar{a}^{3\mu} = 0$$

so that the new conditions are just these three.

$$\begin{aligned} \bar{a}^{00} &= 0 \\ \bar{a}^{10} &= 0 \\ \bar{a}^{20} &= 0 \end{aligned}$$

After the new coordinate variation makes the replacement

$$\bar{a}^{\mu\nu} \rightarrow \bar{a}^{\mu\nu} - ik^\mu \ell^\nu - ik^\nu \ell^\mu + i\eta^{\mu\nu} k_\rho \ell^\rho$$

these conditions become

$$\begin{aligned} \bar{a}^{00} - ik^0\ell^0 - ik^0\ell^0 + i\eta^{00} k_\rho \ell^\rho &= 0 \\ \bar{a}^{10} - ik^0\ell^1 &= 0 \\ \bar{a}^{20} - ik^0\ell^2 &= 0 \end{aligned}$$

The last two can only be solved for ℓ^1 and ℓ^2

$$\ell^1 = -i\bar{a}^{10}/k^0, \qquad \ell^2 = -i\bar{a}^{20}/k^0$$

The first condition becomes

$$\bar{a}^{00} - 2ik^0\ell^0 - ik_0\ell^0 - ik_3\ell^3 = 0$$

or

$$\bar{a}^{00} + ik_0\ell^0 - ik_3\ell^3 = 0$$

or

$$\bar{a}^{00} + i\left(k_0\ell^0 - k_3\ell^3\right) = 0$$

which can be solved for just

$$k_0\ell^0 - k_3\ell^3 = i\bar{a}^{00}$$

There is still one combination of coefficients unfixed, so we can impose one more condition.

$$\bar{a}^{11} + \bar{a}^{22} + \bar{a}^{33} = 0$$

which, after the replacement

$$\bar{a}^{\mu\nu} \to \bar{a}^{\mu\nu} - ik^{\mu}\ell^{\nu} - ik^{\nu}\ell^{\mu} + i\eta^{\mu\nu}k_{\rho}\ell^{\rho}$$

becomes

$$\bar{a}^{11} + \bar{a}^{22} + \bar{a}^{33} - 2ik^3\ell^3 - 3i\left(k_0\ell^0 + k_3\ell^3\right) = 0$$

But

$$k_0\ell^0 = k_3\ell^3 + i\bar{a}^{00}$$

so the resulting condition is

$$\bar{a}^{11} + \bar{a}^{22} + \bar{a}^{33} + 3\bar{a}^{00} - 8ik^3\ell^3 = 0$$

and can be solved for the last coordinate parameter:

$$\ell^3 = -\frac{i}{8}\left(\bar{a}^{11} + \bar{a}^{22} + \bar{a}^{33} + 3\bar{a}^{00}\right)$$

2.1.4 Summary of Radiation Gauge

In terms of the original metric fluctuation field, the radiation gauge imposes the conditions

$$\partial_{\alpha}\bar{h}^{\alpha\beta} = 0$$

$$\bar{h}^{0\mu} = 0$$

and

$$\bar{h} = 0$$

Because all of the time components are set to zero, the conditions can be stated in terms of just the space components

$$\partial_a\bar{h}^{ab} = 0, \qquad \bar{h}^{11} + \bar{h}^{22} + \bar{h}^{33} = 0$$

The first condition states that the spatial metric fluctuation is *transverse* to the direction of propagation while the second says that it is *trace-free*. In a coordinate system aligned with the wave so that ∂_3 is the propagation direction, the non-zero components of the metric fluctuation form a two-by-two symmetric trace-free matrix

$$\begin{bmatrix} \bar{h}_{11} & \bar{h}_{12} \\ \bar{h}_{21} & \bar{h}_{22} \end{bmatrix} = h_+ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + h_\times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

We are left with just two dynamical degrees of freedom or polarizations.

2.2 Effects on Matter

2.2.1 Geodesic Deviation Equation

Recall the relative acceleration of freely falling particles in the form

$$a = K(n)$$

where K is a second rank tensor called the *tidal force tensor* given in terms of the curvature tensor

$$K(n) = \mathcal{R}(u, n)\, u$$

In a local Lorentz frame that is set up around one freely falling particle, the relative acceleration of a neighbor at position

$$n = x^i \partial_i$$

is then

$$\frac{d^2 x^i}{dt^2} = R^i{}_{00j} x^j$$

where t is the time in the local Lorentz frame and x^j are the space coordinates of the neighboring particle in that frame.

Recall the linearized curvature components

$$\delta R_{\sigma\beta\mu\nu} = \frac{1}{2}\left(\partial_\mu \partial_\beta h_{\nu\sigma} - \partial_\mu \partial_\sigma h_{\beta\nu} - \partial_\nu \partial_\beta h_{\mu\sigma} + \partial_\nu \partial_\sigma h_{\beta\mu}\right)$$

so that the curvature in the wave becomes

$$\begin{aligned} R^i{}_{00j} &= R_{i00j} \\ &= \frac{1}{2}\left(\partial_0 \partial_0 h_{ji} - \partial_0 \partial_i h_{0j} - \partial_j \partial_0 h_{0i} + \partial_j \partial_i h_{00}\right) \end{aligned}$$

In the radiation gauge, we just get one surviving term

$$R^i{}_{00j} = \frac{1}{2} \frac{\partial^2}{\partial t^2} h_{ji}$$

so

$$\frac{d^2 x^i}{dt^2} = \frac{1}{2} x^j \left(\frac{\partial^2}{\partial t^2} h_{ji}\right)$$

This equation can be integrated with respect to time. If the neighboring particle is at rest at position $x^i(0)$ at the instant that the gravitational wave hits, then its subsequent position will be

$$x^i(t) = x^i(0) + \frac{1}{2} x^j(0)\, h_{ji}$$

In a coordinate system adapted to the wave, with $x^3 = z$ the propagation direction and $x^1 = x$ and $x^2 = y$ the transverse directions, the particle position will then be

$$\begin{aligned} z(t) &= z(0) \\ x(t) &= x(0) + \frac{1}{2}\left(x(0)\,h_{11} + y(0)\,h_{21}\right) \\ y(t) &= y(0) + \frac{1}{2}\left(x(0)\,h_{12} + y(0)\,h_{22}\right) \end{aligned}$$

Notice that all of the particle motion is perpendicular to the direction of wave propagation, so this really is a transverse wave.

2.2.2 Interparticle Distances and Strain

So far, we just have a description of relative particle motion in terms of local Lorentz coordinates. What does that mean for actual distances? Consider a wave with $h_{21} = 0$ hitting a particle that is on the $x-$axis of the reference particle in this adapted coordinate system. The distance between the particle and the reference is

$$\begin{aligned} d(t) &= x(t)\sqrt{g_{11}} \\ &= x(t)\left(1 + \frac{1}{2}h_{11}\right) \\ &= \left(x(0) + \frac{1}{2}x(0)\,h_{11}\right)\left(1 + \frac{1}{2}h_{11}\right) \\ &= x(0)\left(1 + h_{11}\right) + o\left(h^2\right) \end{aligned}$$

Notice that there are two separate effects. The distance changes because the metric tensor changes the amount of distance that is associated with each coordinate interval. The distance also changes because the coordinate position of the particle changes. Both effects are in the *same* direction and of equal size. This calculation is a *really* good example of how important a plus or minus sign can be. Had the sign turned out the other way, there would be no physical effect at all.

In general, the metric fluctuation h_{ij} becomes the *actual strain* ($\frac{\Delta L}{L}$) that is imposed on objects in the path of the wave. For most instruments, that strain is the quantity that is actually measured and is usually the quantity that is quoted when predicting the strength of a possible gravitational wave. For example, the 1987a supernova in the Large Magellanic Cloud, should have sent out gravitational waves that reached Earth with h_{ij} on the order of 10^{-19}. A Michelson Interferometer with arms 5,000 meters long that functions in the normal way would be able to detect length changes of about 1% of a wavelength of light. For light with a wavelength of 500 nm, the detectable strain would then be 500×10^{-9} m divided by 5000 m or just 10^{-10}. Clearly, a useful Laser Interferometric Gravitational Wave Observatory (such as LIGO) needs to function in a far more sophisticated way than a normal Michelson Interferometer.

2.3 Stress-Energy of a Gravitational Wave

2.3.1 Going to Second Order

Consider a one-parameter family of spacetime metrics $g_{\mu\nu}(\varepsilon)$ with

$$g_{\mu\nu}(0) = \eta_{\mu\nu}$$

and

$$\left.\frac{\partial g_{\mu\nu}}{\partial \varepsilon}\right|_{\varepsilon=0} = h_{\mu\nu}$$

where $h_{\mu\nu}$ is a gravitational wave that solves the linearized Einstein Equations in the radiation gauge.

$$\left.\frac{\partial}{\partial \varepsilon} G^{\mu\nu}\left\{g_{\alpha\beta}(\varepsilon)\right\}\right|_{\varepsilon=0} = G^{\mu\nu}_{\text{lin}}\left\{h_{\alpha\beta}\right\} = 0$$

Here, the curly brackets indicate a functional dependence that may include derivatives and integrals of the argument.

Now refine the solution by adding the next term to the series expansion of the metric.

$$g_{\mu\nu}(\varepsilon) = \eta_{\mu\nu} + \varepsilon h_{\mu\nu} + \varepsilon^2 h^{(2)}_{\mu\nu}$$

and attempt to solve the next order Einstein Equation

$$\left.\frac{\partial^2}{\partial \varepsilon^2} G^{\mu\nu}\left\{g_{\mu\nu}(\varepsilon)\right\}\right|_{\varepsilon=0} = 0$$

for the correction $h^{(2)}_{\mu\nu}$. Notice that we want $h^{(2)}_{\mu\nu}$ to be the correction to the metric for $\varepsilon = 1$ rather than a second derivative of $g_{\mu\nu}(\varepsilon)$ with respect to ε.

Now rearrange the higher order Einstein Equation.

$$\begin{aligned}\left.\frac{\partial^2}{\partial \varepsilon^2} G^{\mu\nu}\left\{g_{\mu\nu}(\varepsilon)\right\}\right|_{\varepsilon=0} &= \left.2\frac{\partial}{\partial s} G^{\mu\nu}\left\{\eta_{\mu\nu} + s h^{(2)}_{\mu\nu}\right\}\right|_{s=0} + \left.\frac{\partial^2}{\partial \varepsilon^2} G^{\mu\nu}\left\{\eta_{\mu\nu} + \varepsilon h_{\mu\nu}\right\}\right|_{\varepsilon=0} \\ &= 2G^{\mu\nu}_{\text{lin}}\left\{h^{(2)}_{\alpha\beta}\right\} + \left.\frac{\partial^2}{\partial \varepsilon^2} G^{\mu\nu}\left\{\eta_{\mu\nu} + \varepsilon h_{\mu\nu}\right\}\right|_{\varepsilon=0}\end{aligned}$$

so we need to solve

$$2G^{\mu\nu}_{\text{lin}}\left\{h^{(2)}_{\alpha\beta}\right\} + \left.\frac{\partial^2}{\partial \varepsilon^2} G^{\mu\nu}\left\{\eta_{\mu\nu} + \varepsilon h_{\mu\nu}\right\}\right|_{\varepsilon=0} = 0$$

Move the second partial derivative term to the right side of the equation

$$G^{\mu\nu}_{\text{lin}}\left\{h^{(2)}_{\alpha\beta}\right\} = -\left.\frac{1}{2}\frac{\partial^2}{\partial \varepsilon^2} G^{\mu\nu}\left\{\eta_{\alpha\beta} + \varepsilon h_{\alpha\beta}\right\}\right|_{\varepsilon=0}$$

Now we have a linearized Einstein Equation

$$G^{\mu\nu}_{\text{lin}}\left\{h^{(2)}_{\alpha\beta}\right\} = 8\pi k T^{\mu\nu}_{\text{eff}}$$

with an effective stress energy tensor

$$T^{\mu\nu}_{\text{eff}} = -\frac{1}{16\pi k}\ \frac{\partial^2}{\partial\varepsilon^2} G^{\mu\nu}\left\{\eta_{\alpha\beta} + \varepsilon h_{\alpha\beta}\right\}\Big|_{\varepsilon=0}$$

due to the gravitational wave.

2.3.2 Calculating Second Order Curvature Components

Now recall that the Riemann tensor is given by

$$R^{\alpha}{}_{\beta\gamma\delta} = \partial_\gamma \Gamma^{\alpha}{}_{\beta\delta} - \partial_\delta \Gamma^{\alpha}{}_{\beta\gamma} + \Gamma^{\alpha}{}_{\sigma\gamma}\Gamma^{\sigma}{}_{\beta\delta} - \Gamma^{\alpha}{}_{\sigma\delta}\Gamma^{\sigma}{}_{\beta\gamma}$$

with

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\sigma}\left(\partial_\gamma g_{\sigma\beta} + \partial_\beta g_{\gamma\sigma} - \partial_\sigma g_{\beta\gamma}\right)$$

Here we will take the metric components to be

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \varepsilon h_{\alpha\beta}$$

and wish to evaluate

$$\frac{\partial^2}{\partial\varepsilon^2} G^{\mu\nu}\left\{\eta_{\alpha\beta} + \varepsilon h_{\alpha\beta}\right\}\Big|_{\varepsilon=0}$$

Start with just the Ricci tensor

$$\frac{\partial^2}{\partial\varepsilon^2} R^{\mu\nu}\left\{\eta_{\alpha\beta} + \varepsilon h_{\alpha\beta}\right\}\Big|_{\varepsilon=0} = \frac{\partial^2}{\partial\varepsilon^2} g^{\mu\beta} g^{\nu\delta} R^{\alpha}{}_{\beta\alpha\delta}\Big|_{\varepsilon=0}$$

For convenience, use dots to denote derivatives with respect to the perturbation parameter ε.

$$\dot{g}_{\alpha\beta} = \frac{\partial g_{\alpha\beta}}{\partial\varepsilon} = h_{\alpha\beta}$$

$$\begin{aligned}
\dot{g}^{\mu\beta} &= -g^{\mu\rho}\dot{g}_{\rho\sigma}g^{\sigma\beta} = -h^{\mu\beta} \\
\ddot{g}^{\mu\beta} &= -\dot{g}^{\mu\rho}\dot{g}_{\rho\sigma}g^{\sigma\beta} - g^{\mu\rho}\dot{g}_{\rho\sigma}\dot{g}^{\sigma\beta} \\
&= h^{\mu\rho}h_{\rho}{}^{\beta} + h^{\mu}{}_{\sigma}h^{\sigma\beta} \\
&= 2h^{\mu}{}_{\sigma}h^{\sigma\beta}
\end{aligned}$$

$$\begin{aligned}
\dot{\Gamma}^{\alpha}{}_{\beta\gamma} &= \frac{1}{2}\dot{g}^{\alpha\sigma}\left(\partial_\gamma g_{\sigma\beta} + \partial_\beta g_{\gamma\sigma} - \partial_\sigma g_{\beta\gamma}\right) + \frac{1}{2}g^{\alpha\sigma}\left(\partial_\gamma h_{\sigma\beta} + \partial_\beta h_{\gamma\sigma} - \partial_\sigma h_{\beta\gamma}\right) \\
\ddot{\Gamma}^{\alpha}{}_{\beta\gamma} &= \frac{1}{2}\ddot{g}^{\alpha\sigma}\left(\partial_\gamma g_{\sigma\beta} + \partial_\beta g_{\gamma\sigma} - \partial_\sigma g_{\beta\gamma}\right) + \frac{1}{2}\dot{g}^{\alpha\sigma}\left(\partial_\gamma h_{\sigma\beta} + \partial_\beta h_{\gamma\sigma} - \partial_\sigma h_{\beta\gamma}\right) + \frac{1}{2}\dot{g}^{\alpha\sigma}\left(\partial_\gamma h_{\sigma\beta} + \partial_\beta h_{\gamma\sigma} - \partial_\sigma h_{\beta\gamma}\right) \\
&= h^{\alpha}{}_{\lambda}h^{\lambda\sigma}\left(\partial_\gamma g_{\sigma\beta} + \partial_\beta g_{\gamma\sigma} - \partial_\sigma g_{\beta\gamma}\right) - h^{\alpha\sigma}\left(\partial_\gamma h_{\sigma\beta} + \partial_\beta h_{\gamma\sigma} - \partial_\sigma h_{\beta\gamma}\right)
\end{aligned}$$

$$\ddot{\Gamma}^{\alpha}{}_{\beta\gamma}\Big|_{\varepsilon=0} = -h^{\alpha\sigma}\left(\partial_\gamma h_{\sigma\beta} + \partial_\beta h_{\gamma\sigma} - \partial_\sigma h_{\beta\gamma}\right)$$

$$\dot{R}^{\alpha}{}_{\beta\alpha\delta} = \partial_\alpha \dot{\Gamma}^{\alpha}{}_{\beta\delta} - \partial_\delta \dot{\Gamma}^{\alpha}{}_{\beta\alpha} + \dot{\Gamma}^{\alpha}{}_{\sigma\alpha}\Gamma^{\sigma}{}_{\beta\delta} + \Gamma^{\alpha}{}_{\sigma\alpha}\dot{\Gamma}^{\sigma}{}_{\beta\delta} - \dot{\Gamma}^{\alpha}{}_{\sigma\delta}\Gamma^{\sigma}{}_{\beta\alpha} - \Gamma^{\alpha}{}_{\sigma\delta}\dot{\Gamma}^{\sigma}{}_{\beta\alpha}$$

$$\begin{aligned}
\ddot{R}^{\alpha}{}_{\beta\alpha\delta} &= \partial_\alpha \ddot{\Gamma}^{\alpha}{}_{\beta\delta} - \partial_\delta \ddot{\Gamma}^{\alpha}{}_{\beta\alpha} \\
&\quad + \ddot{\Gamma}^{\alpha}{}_{\sigma\alpha}\Gamma^{\sigma}{}_{\beta\delta} + \dot{\Gamma}^{\alpha}{}_{\sigma\alpha}\dot{\Gamma}^{\sigma}{}_{\beta\delta} + \dot{\Gamma}^{\alpha}{}_{\sigma\alpha}\dot{\Gamma}^{\sigma}{}_{\beta\delta} + \Gamma^{\alpha}{}_{\sigma\alpha}\ddot{\Gamma}^{\sigma}{}_{\beta\delta} \\
&\quad - \ddot{\Gamma}^{\alpha}{}_{\sigma\delta}\Gamma^{\sigma}{}_{\beta\alpha} - \dot{\Gamma}^{\alpha}{}_{\sigma\delta}\dot{\Gamma}^{\sigma}{}_{\beta\alpha} - \dot{\Gamma}^{\alpha}{}_{\sigma\delta}\dot{\Gamma}^{\sigma}{}_{\beta\alpha} - \Gamma^{\alpha}{}_{\sigma\delta}\ddot{\Gamma}^{\sigma}{}_{\beta\alpha} \\
&= \partial_\alpha \ddot{\Gamma}^{\alpha}{}_{\beta\delta} - \partial_\delta \ddot{\Gamma}^{\alpha}{}_{\beta\alpha} + \ddot{\Gamma}^{\alpha}{}_{\sigma\alpha}\Gamma^{\sigma}{}_{\beta\delta} + \Gamma^{\alpha}{}_{\sigma\alpha}\ddot{\Gamma}^{\sigma}{}_{\beta\delta} - \ddot{\Gamma}^{\alpha}{}_{\sigma\delta}\Gamma^{\sigma}{}_{\beta\alpha} - \Gamma^{\alpha}{}_{\sigma\delta}\ddot{\Gamma}^{\sigma}{}_{\beta\alpha} \\
&\quad + 2\dot{\Gamma}^{\alpha}{}_{\sigma\alpha}\dot{\Gamma}^{\sigma}{}_{\beta\delta} - 2\dot{\Gamma}^{\alpha}{}_{\sigma\delta}\dot{\Gamma}^{\sigma}{}_{\beta\alpha}
\end{aligned}$$

$$\frac{\partial}{\partial\varepsilon}\left(g^{\mu\beta}g^{\nu\delta}R^{\alpha}{}_{\beta\alpha\delta}\right) = \dot{g}^{\mu\beta}g^{\nu\delta}R^{\alpha}{}_{\beta\alpha\delta} + g^{\mu\beta}\dot{g}^{\nu\delta}R^{\alpha}{}_{\beta\alpha\delta} + g^{\mu\beta}g^{\nu\delta}\dot{R}^{\alpha}{}_{\beta\alpha\delta}$$

Now take another derivative, but discard any term that will give zero for $\varepsilon = 0$.

$$\begin{aligned}
\left.\frac{\partial^2}{\partial\varepsilon^2}\left(g^{\mu\beta}g^{\nu\delta}R^{\alpha}{}_{\beta\alpha\delta}\right)\right|_{\varepsilon=0} &= \dot{g}^{\mu\beta}g^{\nu\delta}\dot{R}^{\alpha}{}_{\beta\alpha\delta} + g^{\mu\beta}\dot{g}^{\nu\delta}\dot{R}^{\alpha}{}_{\beta\alpha\delta} \\
&\quad + \dot{g}^{\mu\beta}g^{\nu\delta}\dot{R}^{\alpha}{}_{\beta\alpha\delta} + g^{\mu\beta}\dot{g}^{\nu\delta}\dot{R}^{\alpha}{}_{\beta\alpha\delta} + g^{\mu\beta}g^{\nu\delta}\ddot{R}^{\alpha}{}_{\beta\alpha\delta} \\
&= \left.\left\{2\left(h^{\mu\beta}g^{\nu\delta} + g^{\mu\beta}h^{\nu\delta}\right)\dot{R}^{\alpha}{}_{\beta\alpha\delta} + g^{\mu\beta}g^{\nu\delta}\ddot{R}^{\alpha}{}_{\beta\alpha\delta}\right\}\right|_{\varepsilon=0}
\end{aligned}$$

Since the connection coefficients vanish for $\varepsilon = 0$ there are some simplifications

$$\begin{aligned}
\left.\dot{R}^{\alpha}{}_{\beta\alpha\delta}\right|_{\varepsilon=0} &= \partial_\alpha \dot{\Gamma}^{\alpha}{}_{\beta\delta} - \partial_\delta \dot{\Gamma}^{\alpha}{}_{\beta\alpha} \\
\left.\ddot{R}^{\alpha}{}_{\beta\alpha\delta}\right|_{\varepsilon=0} &= \partial_\alpha \ddot{\Gamma}^{\alpha}{}_{\beta\delta} - \partial_\delta \ddot{\Gamma}^{\alpha}{}_{\beta\alpha} + 2\dot{\Gamma}^{\alpha}{}_{\sigma\alpha}\dot{\Gamma}^{\sigma}{}_{\beta\delta} - 2\dot{\Gamma}^{\alpha}{}_{\sigma\delta}\dot{\Gamma}^{\sigma}{}_{\beta\alpha}
\end{aligned}$$

$$\begin{aligned}
\left.\frac{\partial^2}{\partial\varepsilon^2}\left(g^{\mu\beta}g^{\nu\delta}R^{\alpha}{}_{\beta\alpha\delta}\right)\right|_{\varepsilon=0} &= 2\left(h^{\mu\beta}g^{\nu\delta} + g^{\mu\beta}h^{\nu\delta}\right)\left(\partial_\alpha \dot{\Gamma}^{\alpha}{}_{\beta\delta} - \partial_\delta \dot{\Gamma}^{\alpha}{}_{\beta\alpha}\right) \\
&\quad + g^{\mu\beta}g^{\nu\delta}\left(\partial_\alpha \ddot{\Gamma}^{\alpha}{}_{\beta\delta} - \partial_\delta \ddot{\Gamma}^{\alpha}{}_{\beta\alpha} + 2\dot{\Gamma}^{\alpha}{}_{\sigma\alpha}\dot{\Gamma}^{\sigma}{}_{\beta\delta} - 2\dot{\Gamma}^{\alpha}{}_{\sigma\delta}\dot{\Gamma}^{\sigma}{}_{\beta\alpha}\right)
\end{aligned}$$

For $\varepsilon = 0$

$$\begin{aligned}
\dot{\Gamma}^{\alpha}{}_{\beta\delta} &= \frac{1}{2}\eta^{\alpha\sigma}\left(\partial_\delta h_{\sigma\beta} + \partial_\beta h_{\delta\sigma} - \partial_\sigma h_{\beta\delta}\right) \\
\dot{\Gamma}^{\alpha}{}_{\beta\alpha} &= \frac{1}{2}\eta^{\alpha\sigma}\left(\partial_\alpha h_{\sigma\beta} + \partial_\beta h_{\alpha\sigma} - \partial_\sigma h_{\beta\alpha}\right)
\end{aligned}$$

For waves in the radiation gauge, all of the terms in the second expression are zero, so

$$\dot{\Gamma}^{\alpha}{}_{\beta\alpha} = 0$$

$$\begin{aligned}
\partial_\alpha \dot{\Gamma}^{\alpha}{}_{\beta\delta} - \partial_\delta \dot{\Gamma}^{\alpha}{}_{\beta\alpha} &= \frac{1}{2}\eta^{\alpha\sigma}\left(\partial_\alpha\partial_\delta h_{\sigma\beta} + \partial_\alpha\partial_\beta h_{\delta\sigma} - \partial_\alpha\partial_\sigma h_{\beta\delta} - \partial_\delta\partial_\alpha h_{\sigma\beta} - \partial_\delta\partial_\beta h_{\alpha\sigma} + \partial_\delta\partial_\sigma h_{\beta\alpha}\right) \\
&= \frac{1}{2}\eta^{\alpha\sigma}\left(\partial_\alpha\partial_\beta h_{\delta\sigma} - \partial_\alpha\partial_\sigma h_{\beta\delta} - \partial_\delta\partial_\beta h_{\alpha\sigma} + \partial_\delta\partial_\sigma h_{\beta\alpha}\right) \\
&= \frac{1}{2}\left(\partial_\alpha\partial_\beta h_\delta{}^{\alpha} - \Box h_{\beta\delta} - \partial_\delta\partial_\beta h + \partial_\delta\partial_\sigma h_\beta{}^{\sigma}\right) \\
&= \frac{1}{2}\left(\partial_\beta\left(\partial_\alpha h_\delta{}^{\alpha}\right) + \partial_\delta\left(\partial_\alpha h_\beta{}^{\alpha}\right) - \partial_\delta\partial_\beta h - \Box h_{\beta\delta}\right)
\end{aligned}$$

For waves in the radiation gauge, all of these terms are zero, so

$$\partial_\alpha \dot{\Gamma}^\alpha{}_{\beta\delta} - \partial_\delta \dot{\Gamma}^\alpha{}_{\beta\alpha} = 0$$

$$\begin{aligned}
\partial_\alpha \ddot{\Gamma}^\alpha{}_{\beta\delta} - \partial_\delta \ddot{\Gamma}^\alpha{}_{\beta\alpha} &= -\partial_\alpha \left(h^{\alpha\sigma} \left(\partial_\delta h_{\sigma\beta} + \partial_\beta h_{\delta\sigma} - \partial_\sigma h_{\beta\delta} \right) \right) + \partial_\delta \left(h^{\alpha\sigma} \left(\partial_\alpha h_{\sigma\beta} + \partial_\beta h_{\alpha\sigma} - \partial_\sigma h_{\beta\alpha} \right) \right) \\
&= -\left(\partial_\alpha h^{\alpha\sigma} \right) \left(\partial_\delta h_{\sigma\beta} + \partial_\beta h_{\delta\sigma} - \partial_\sigma h_{\beta\delta} \right) - h^{\alpha\sigma} \left(\partial_\delta \partial_\alpha h_{\sigma\beta} + \partial_\beta \partial_\alpha h_{\delta\sigma} - \partial_\sigma \partial_\alpha h_{\beta\delta} \right) \\
&= -\left(\partial_\alpha h^{\alpha\sigma} \right) \left(\partial_\delta h_{\sigma\beta} \right) - \left(\partial_\alpha h^{\alpha\sigma} \right) \left(\partial_\beta h_{\delta\sigma} \right) + \left(\partial_\alpha h^{\alpha\sigma} \right) \left(\partial_\sigma h_{\beta\delta} \right) \\
&\quad -h^{\alpha\sigma} \left(\partial_\delta \partial_\alpha h_{\sigma\beta} + \partial_\beta \partial_\alpha h_{\delta\sigma} - \partial_\sigma \partial_\alpha h_{\beta\delta} \right)
\end{aligned}$$

or, using the gauge conditions,

$$\partial_\alpha \ddot{\Gamma}^\alpha{}_{\beta\delta} - \partial_\delta \ddot{\Gamma}^\alpha{}_{\beta\alpha} = -h^{\alpha\sigma} \left(\partial_\delta \partial_\alpha h_{\sigma\beta} + \partial_\beta \partial_\alpha h_{\delta\sigma} - \partial_\sigma \partial_\alpha h_{\beta\delta} \right)$$

Now assemble what we have so far:

$$\begin{aligned}
\frac{\partial^2}{\partial\varepsilon^2} \left(g^{\mu\beta} g^{\nu\delta} R^\alpha{}_{\beta\alpha\delta} \right) \Big|_{\varepsilon=0} &= g^{\mu\beta} g^{\nu\delta} \left(\partial_\alpha \ddot{\Gamma}^\alpha{}_{\beta\delta} - \partial_\delta \ddot{\Gamma}^\alpha{}_{\beta\alpha} + 2\dot{\Gamma}^\alpha{}_{\sigma\alpha} \dot{\Gamma}^\sigma{}_{\beta\delta} - 2\dot{\Gamma}^\alpha{}_{\sigma\delta} \dot{\Gamma}^\sigma{}_{\beta\alpha} \right) \\
&= g^{\mu\beta} g^{\nu\delta} \left(\partial_\alpha \ddot{\Gamma}^\alpha{}_{\beta\delta} - \partial_\delta \ddot{\Gamma}^\alpha{}_{\beta\alpha} \right) \\
&\quad +2g^{\mu\beta} g^{\nu\delta} \dot{\Gamma}^\alpha{}_{\sigma\alpha} \dot{\Gamma}^\sigma{}_{\beta\delta} - 2g^{\mu\beta} g^{\nu\delta} \dot{\Gamma}^\alpha{}_{\sigma\delta} \dot{\Gamma}^\sigma{}_{\beta\alpha} \\
&= g^{\mu\beta} g^{\nu\delta} \left[-h^{\alpha\sigma} \left(\partial_\delta \partial_\alpha h_{\sigma\beta} + \partial_\beta \partial_\alpha h_{\delta\sigma} - \partial_\sigma \partial_\alpha h_{\beta\delta} \right) \right] \\
&\quad +2g^{\mu\beta} g^{\nu\delta} \dot{\Gamma}^\alpha{}_{\sigma\alpha} \dot{\Gamma}^\sigma{}_{\beta\delta} - 2g^{\mu\beta} g^{\nu\delta} \dot{\Gamma}^\alpha{}_{\sigma\delta} \dot{\Gamma}^\sigma{}_{\beta\alpha}
\end{aligned}$$

Notice that the contracted connection coefficient is zero for radiation gauge waves so that

$$\frac{\partial^2}{\partial\varepsilon^2} \left(g^{\mu\beta} g^{\nu\delta} R^\alpha{}_{\beta\alpha\delta} \right) \Big|_{\varepsilon=0} = g^{\mu\beta} g^{\nu\delta} \left[-h^{\alpha\sigma} \left(\partial_\delta \partial_\alpha h_{\sigma\beta} + \partial_\beta \partial_\alpha h_{\delta\sigma} - \partial_\sigma \partial_\alpha h_{\beta\delta} \right) - 2\dot{\Gamma}^\alpha{}_{\sigma\delta} \dot{\Gamma}^\sigma{}_{\beta\alpha} \right]$$

or

$$\begin{aligned}
\frac{\partial^2}{\partial\varepsilon^2} \left(g^{\mu\beta} g^{\nu\delta} R^\alpha{}_{\beta\alpha\delta} \right) \Big|_{\varepsilon=0} &= -\eta^{\nu\delta} h^{\alpha\sigma} \partial_\delta \partial_\alpha h^\mu{}_\sigma - \eta^{\mu\beta} h^{\alpha\sigma} \partial_\beta \partial_\alpha h^\nu{}_\sigma - h^{\alpha\sigma} \partial_\sigma \partial_\alpha h^{\mu\nu} \\
&\quad -2\eta^{\mu\beta} \eta^{\nu\delta} \dot{\Gamma}^\alpha{}_{\sigma\delta} \dot{\Gamma}^\sigma{}_{\beta\alpha}
\end{aligned}$$

That leaves the product of connection coeficient variations to work out.

$$\begin{aligned}
\dot{\Gamma}^\alpha{}_{\sigma\delta} &= \frac{1}{2} \eta^{\alpha\rho} \left(\partial_\delta h_{\rho\sigma} + \partial_\sigma h_{\delta\rho} - \partial_\rho h_{\sigma\delta} \right) \\
\dot{\Gamma}^\sigma{}_{\beta\alpha} &= \frac{1}{2} \eta^{\sigma\lambda} \left(\partial_\alpha h_{\lambda\beta} + \partial_\beta h_{\alpha\lambda} - \partial_\lambda h_{\beta\alpha} \right)
\end{aligned}$$

$$\begin{aligned}
2\dot{\Gamma}^\alpha{}_{\sigma\delta} \dot{\Gamma}^\sigma{}_{\beta\alpha} &= \frac{1}{2} \eta^{\alpha\rho} \eta^{\sigma\lambda} \left(\partial_\delta h_{\rho\sigma} + \partial_\sigma h_{\delta\rho} - \partial_\rho h_{\sigma\delta} \right) \left(\partial_\alpha h_{\lambda\beta} + \partial_\beta h_{\alpha\lambda} - \partial_\lambda h_{\beta\alpha} \right) \\
4\dot{\Gamma}^\alpha{}_{\sigma\delta} \dot{\Gamma}^\sigma{}_{\beta\alpha} &= \eta^{\alpha\rho} \eta^{\sigma\lambda} \partial_\delta h_{\rho\sigma} \partial_\alpha h_{\lambda\beta} + \eta^{\alpha\rho} \eta^{\sigma\lambda} \partial_\delta h_{\rho\sigma} \partial_\beta h_{\alpha\lambda} - \eta^{\alpha\rho} \eta^{\sigma\lambda} \partial_\delta h_{\rho\sigma} \partial_\lambda h_{\beta\alpha} \\
&\quad +\eta^{\alpha\rho} \eta^{\sigma\lambda} \partial_\sigma h_{\delta\rho} \partial_\alpha h_{\lambda\beta} + \eta^{\alpha\rho} \eta^{\sigma\lambda} \partial_\sigma h_{\delta\rho} \partial_\beta h_{\alpha\lambda} - \eta^{\alpha\rho} \eta^{\sigma\lambda} \partial_\sigma h_{\delta\rho} \partial_\lambda h_{\beta\alpha} \\
&\quad -\eta^{\alpha\rho} \eta^{\sigma\lambda} \partial_\rho h_{\sigma\delta} \partial_\alpha h_{\lambda\beta} - \eta^{\alpha\rho} \eta^{\sigma\lambda} \partial_\rho h_{\sigma\delta} \partial_\beta h_{\alpha\lambda} + \eta^{\alpha\rho} \eta^{\sigma\lambda} \partial_\rho h_{\sigma\delta} \partial_\lambda h_{\beta\alpha}
\end{aligned}$$

Reduce the number of indexes by using the metric to raise indexes wherever possible.

$$\begin{aligned}
4\dot{\Gamma}^{\alpha}{}_{\sigma\delta}\dot{\Gamma}^{\sigma}{}_{\beta\alpha} &= \partial_\delta h^{\alpha\lambda}\partial_\alpha h_{\lambda\beta} + \partial_\delta h^{\alpha\lambda}\partial_\beta h_{\alpha\lambda} - \partial_\delta h^{\alpha\lambda}\partial_\lambda h_{\beta\alpha} \\
&\quad + \partial_\sigma h_\delta{}^\alpha \partial_\alpha h^\sigma{}_\beta + \partial_\sigma h_\delta{}^\alpha \partial_\beta h_\alpha{}^\sigma - \eta^{\sigma\lambda}\partial_\sigma h_\delta{}^\alpha \partial_\lambda h_{\beta\alpha} \\
&\quad - \eta^{\alpha\rho}\partial_\rho h^\lambda{}_\delta \partial_\alpha h_{\lambda\beta} - \partial_\rho h^\lambda{}_\delta \partial_\beta h^\rho{}_\lambda + \partial_\rho h^\lambda{}_\delta \partial_\lambda h^\rho{}_\beta
\end{aligned}$$

Match up like terms.

$$\begin{aligned}
4\dot{\Gamma}^{\alpha}{}_{\sigma\delta}\dot{\Gamma}^{\sigma}{}_{\beta\alpha} &= \partial_\delta h^{\alpha\lambda}\partial_\alpha h_{\lambda\beta} - \partial_\delta h^{\alpha\lambda}\partial_\lambda h_{\beta\alpha} - \eta^{\sigma\lambda}\partial_\sigma h_\delta{}^\alpha \partial_\lambda h_{\beta\alpha} - \eta^{\alpha\rho}\partial_\rho h^\lambda{}_\delta \partial_\alpha h_{\lambda\beta} \\
&\quad + \partial_\delta h^{\alpha\lambda}\partial_\beta h_{\alpha\lambda} - \partial_\rho h^\lambda{}_\delta \partial_\beta h^\rho{}_\lambda + \partial_\sigma h_\delta{}^\alpha \partial_\beta h_\alpha{}^\sigma \\
&\quad + \partial_\sigma h_\delta{}^\alpha \partial_\alpha h^\sigma{}_\beta + \partial_\rho h^\lambda{}_\delta \partial_\lambda h^\rho{}_\beta \\
&= -2\eta^{\sigma\lambda}\partial_\sigma h_\delta{}^\alpha \partial_\lambda h_{\beta\alpha} + \partial_\delta h^{\alpha\lambda}\partial_\beta h_{\alpha\lambda} + 2\partial_\rho h^\lambda{}_\delta \partial_\lambda h^\rho{}_\beta
\end{aligned}$$

$$\begin{aligned}
-2\eta^{\mu\beta}\eta^{\nu\delta}\dot{\Gamma}^{\alpha}{}_{\sigma\delta}\dot{\Gamma}^{\sigma}{}_{\beta\alpha} &= \eta^{\mu\beta}\eta^{\nu\delta}\eta^{\sigma\lambda}\partial_\sigma h_\delta{}^\alpha \partial_\lambda h_{\beta\alpha} - \eta^{\mu\beta}\eta^{\nu\delta}\partial_\rho h^\lambda{}_\delta \partial_\lambda h^\rho{}_\beta - \frac{1}{2}\eta^{\mu\beta}\eta^{\nu\delta}\partial_\delta h^{\alpha\lambda}\partial_\beta h_{\alpha\lambda} \\
&= \eta^{\sigma\lambda}\partial_\sigma h^{\nu\alpha}\partial_\lambda h^\mu{}_\alpha - \partial_\rho h^{\lambda\nu}\partial_\lambda h^{\rho\mu} - \frac{1}{2}\eta^{\mu\beta}\eta^{\nu\delta}\partial_\delta h^{\alpha\lambda}\partial_\beta h_{\alpha\lambda}
\end{aligned}$$

so that

$$\begin{aligned}
\frac{\partial^2}{\partial\varepsilon^2}\left(g^{\mu\beta}g^{\nu\delta}R^\alpha{}_{\beta\alpha\delta}\right)\Big|_{\varepsilon=0} &= -\eta^{\nu\delta}h^{\alpha\sigma}\partial_\delta\partial_\alpha h^\mu{}_\sigma - \eta^{\mu\beta}h^{\alpha\sigma}\partial_\beta\partial_\alpha h^\nu{}_\sigma - h^{\alpha\sigma}\partial_\sigma\partial_\alpha h^{\mu\nu} \\
&\quad + \eta^{\sigma\lambda}\partial_\sigma h^{\nu\alpha}\partial_\lambda h^\mu{}_\alpha - \partial_\rho h^{\lambda\nu}\partial_\lambda h^{\rho\mu} - \frac{1}{2}\eta^{\mu\beta}\eta^{\nu\delta}\partial_\delta h^{\alpha\lambda}\partial_\beta h_{\alpha\lambda}
\end{aligned}$$

2.3.3 The Short Wavelength Approximation

If we solve the second order system in detail, we will gain some useless information at great cost along with the information that we really need. We will obtain the nonlinear corrections to the rapidly fluctuationg gravitational wave amplitudes. Since these amplitudes are typically around 10^{-19} the nonlinear corrections to them are of no interest at all. To get rid of this useless information, perform a space-time average over a region that is long in comparison to the period of the waves and larger in comparison to their wavelength and solve the corresponding time averaged second order equations:

$$G^{\mu\nu}_{\text{lin}}\left\{\left\langle h^{(2)}_{\alpha\beta}\right\rangle\right\} = \langle T^{\mu\nu}_{\text{eff}}\rangle$$

The time-averaged effective stress energy tensor is actually what we are interested in. It provides the average rate at which the gravitational wave transfers energy and momentum.

The key idea for simplifying the effective stress energy is that any derivative of a rapidly varying function f will average to zero, so

$$\langle\partial_\sigma f\rangle \to 0$$

As a result, we can "integrate by parts" to obtain

$$\begin{aligned} & \left\langle -\eta^{\nu\delta} h^{\alpha\sigma} \partial_\delta \partial_\alpha h^\mu{}_\sigma - \eta^{\mu\beta} h^{\alpha\sigma} \partial_\beta \partial_\alpha h^\nu{}_\sigma - h^{\alpha\sigma} \partial_\sigma \partial_\alpha h^{\mu\nu} \right\rangle \\ = & \ \eta^{\nu\delta} \partial_\alpha h^{\alpha\sigma} \partial_\delta h^\mu{}_\sigma + \eta^{\mu\beta} \partial_\alpha h^{\alpha\sigma} \partial_\beta h^\nu{}_\sigma + \partial_\alpha h^{\alpha\sigma} \partial_\sigma h^{\mu\nu} \end{aligned}$$

which vanishes in the radiation gauge and leaves just

$$\left\langle \frac{\partial^2}{\partial\varepsilon^2} \left(g^{\mu\beta} g^{\nu\delta} R^\alpha{}_{\beta\alpha\delta} \right) \Big|_{\varepsilon=0} \right\rangle = \left\langle \eta^{\sigma\lambda} \partial_\sigma h^{\nu\alpha} \partial_\lambda h^\mu{}_\alpha - \partial_\rho h^{\lambda\nu} \partial_\lambda h^{\rho\mu} - \frac{1}{2} \eta^{\mu\beta} \eta^{\nu\delta} \partial_\delta h^{\alpha\lambda} \partial_\beta h_{\alpha\lambda} \right\rangle$$

The first two terms vanish after integration by parts and leave just

$$\left\langle \frac{\partial^2}{\partial\varepsilon^2} \left(g^{\mu\beta} g^{\nu\delta} R^\alpha{}_{\beta\alpha\delta} \right) \Big|_{\varepsilon=0} \right\rangle = -\frac{1}{2} \eta^{\mu\beta} \eta^{\nu\delta} \left\langle \partial_\delta h^{\alpha\lambda} \partial_\beta h_{\alpha\lambda} \right\rangle$$

Now figure out what happens to the trace

$$\begin{aligned} \left\langle \frac{\partial^2}{\partial\varepsilon^2} g_{\mu\nu} \left(g^{\mu\beta} g^{\nu\delta} R^\alpha{}_{\beta\alpha\delta} \right) \Big|_{\varepsilon=0} \right\rangle &= \left\langle g_{\mu\nu} \frac{\partial^2}{\partial\varepsilon^2} \left(g^{\mu\beta} g^{\nu\delta} R^\alpha{}_{\beta\alpha\delta} \right) \Big|_{\varepsilon=0} \right\rangle \\ & \quad + \left\langle 2\dot{g}_{\mu\nu} \frac{\partial}{\partial\varepsilon} \left(g^{\mu\beta} g^{\nu\delta} R^\alpha{}_{\beta\alpha\delta} \right) \Big|_{\varepsilon=0} \right\rangle \\ &= -\frac{1}{2} \eta_{\mu\nu} \eta^{\mu\beta} \eta^{\nu\delta} \left\langle \partial_\delta h^{\alpha\lambda} \partial_\beta h_{\alpha\lambda} \right\rangle \\ & \quad +2 \left\langle h_{\mu\nu} \ \eta^{\mu\beta} \eta^{\nu\delta} \dot{R}^\alpha{}_{\beta\alpha\delta} \Big|_{\varepsilon=0} \right\rangle \end{aligned}$$

The first term vanishes after an integration by parts because of the wave equation. For the second term, the only part of $\dot{R}^\alpha{}_{\beta\alpha\delta}$ that is non-zero here is the derivative term

$$\dot{R}^\alpha{}_{\beta\alpha\delta} = \partial_\alpha \dot{\Gamma}^\alpha{}_{\beta\delta} - \partial_\delta \dot{\Gamma}^\alpha{}_{\beta\alpha}$$

so

$$\begin{aligned} 2 \left\langle h_{\mu\nu} \ \eta^{\mu\beta} \eta^{\nu\delta} \dot{R}^\alpha{}_{\beta\alpha\delta} \Big|_{\varepsilon=0} \right\rangle &= 2\eta^{\mu\beta} \eta^{\nu\delta} \left\langle h_{\mu\nu} \left(\partial_\alpha \dot{\Gamma}^\alpha{}_{\beta\delta} - \partial_\delta \dot{\Gamma}^\alpha{}_{\beta\alpha} \right) \right\rangle \\ &= 2\eta^{\mu\beta} \eta^{\nu\delta} \left\langle \partial_\delta h_{\mu\nu} \dot{\Gamma}^\alpha{}_{\beta\alpha} - \partial_\alpha h_{\mu\nu} \dot{\Gamma}^\alpha{}_{\beta\delta} \right\rangle \\ &= 2 \left\langle \partial_\delta h^{\beta\delta} \dot{\Gamma}^\alpha{}_{\beta\alpha} - \partial_\alpha h^{\beta\delta} \dot{\Gamma}^\alpha{}_{\beta\delta} \right\rangle \\ &= -2 \left\langle \partial_\alpha h^{\beta\delta} \dot{\Gamma}^\alpha{}_{\beta\delta} \right\rangle \end{aligned}$$

where the gauge condition was used to eliminate a term. Now use

$$\dot{\Gamma}^\alpha{}_{\beta\delta} = \frac{1}{2} \eta^{\alpha\sigma} \left(\partial_\delta h_{\sigma\beta} + \partial_\beta h_{\delta\sigma} - \partial_\sigma h_{\beta\delta} \right)$$

to obtain

$$\begin{aligned} 2 \left\langle h_{\mu\nu} \ \eta^{\mu\beta} \eta^{\nu\delta} \dot{R}^\alpha{}_{\beta\alpha\delta} \Big|_{\varepsilon=0} \right\rangle &= -\eta^{\alpha\sigma} \left\langle \partial_\alpha h^{\beta\delta} \left(\partial_\delta h_{\sigma\beta} + \partial_\beta h_{\delta\sigma} - \partial_\sigma h_{\beta\delta} \right) \right\rangle \\ &= -\left\langle \partial_\alpha h^{\beta\delta} \partial_\delta h^\alpha{}_\beta \right\rangle - \left\langle \partial_\alpha h^{\beta\delta} \partial_\beta h^\alpha{}_\delta \right\rangle + \eta^{\alpha\sigma} \left\langle \partial_\alpha h^{\beta\delta} \partial_\sigma h_{\beta\delta} \right\rangle \end{aligned}$$

and notice that all of these terms vanish after integration by parts.

Now we can get the final result:

$$
\begin{aligned}
\langle T_{\text{eff}}^{\mu\nu} \rangle &= -\frac{1}{16\pi k} \left\langle \frac{\partial^2}{\partial \varepsilon^2} G^{\mu\nu} \left\{ \eta_{\alpha\beta} + \varepsilon h_{\alpha\beta} \right\} \Big|_{\varepsilon=0} \right\rangle \\
&= -\frac{1}{16\pi k} \left\langle \frac{\partial^2}{\partial \varepsilon^2} R^{\mu\nu} \left\{ \eta_{\alpha\beta} + \varepsilon h_{\alpha\beta} \right\} \Big|_{\varepsilon=0} \right\rangle \\
&= \left(-\frac{1}{16\pi k} \right) \left(-\frac{1}{2} \eta^{\mu\beta} \eta^{\nu\delta} \left\langle \partial_\delta h^{\alpha\lambda} \partial_\beta h_{\alpha\lambda} \right\rangle \right)
\end{aligned}
$$

or

$$
\langle T_{\text{eff}}^{\mu\nu} \rangle = \frac{1}{32\pi k} \eta^{\mu\beta} \eta^{\nu\delta} \left\langle \partial_\delta h^{\alpha\lambda} \partial_\beta h_{\alpha\lambda} \right\rangle
$$

Insert the adapted frame plane wave expansion in the form

$$
\begin{aligned}
h^{\mu\nu}(k,x) &= \bar{h}^{\mu\nu}(k,x) = \operatorname{Re} \left\{ \bar{a}^{\mu\nu} e^{ik_\rho x^\rho} \right\} \\
&= \frac{1}{2} \left(\bar{a}^{\mu\nu} e^{ik_\rho x^\rho} + \bar{a}^{*\mu\nu} e^{-ik_\rho x^\rho} \right) \\
&= \frac{1}{2} \left(\bar{a}^{\mu\nu} e^{i\omega\left(x^3 - x^0\right)} + \bar{a}^{*\mu\nu} e^{-i\omega\left(x^3 - x^0\right)} \right)
\end{aligned}
$$

or, for the non-zero components,

$$
\begin{aligned}
\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} &= \frac{1}{2} \left(a_+ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} e^{i\omega\left(x^3 - x^0\right)} + a_+^* \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} e^{-i\omega\left(x^3 - x^0\right)} \right) \\
&+ \frac{1}{2} \left(a_\times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} e^{i\omega\left(x^3 - x^0\right)} + a_\times^* \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} e^{-i\omega\left(x^3 - x^0\right)} \right)
\end{aligned}
$$

The polarization matrices are orthogonal and the non-zero stress-energy components are the non-oscillating terms in the product, so we find just

$$
\begin{aligned}
\left\langle T_{\text{eff}}^{00} \right\rangle &= \frac{1}{32\pi k} \left\langle \partial_0 h^{\alpha\lambda} \partial_0 h_{\alpha\lambda} \right\rangle \\
&= \frac{1}{32\pi k} \left((-i\omega)(+i\omega) a_+ a_+^* + (-i\omega)(+i\omega) a_\times a_\times^* \right) \\
&= \frac{1}{32\pi k} \omega^2 \left(a_+ a_+^* + a_\times a_\times^* \right)
\end{aligned}
$$

$$
\begin{aligned}
\left\langle T_{\text{eff}}^{03} \right\rangle &= -\frac{1}{32\pi k} \left\langle \partial_0 h^{\alpha\lambda} \partial_3 h_{\alpha\lambda} \right\rangle \\
&= -\frac{1}{32\pi k} \left((-i\omega)(-i\omega) a_+ a_+^* + (-i\omega)(-i\omega) a_\times a_\times^* \right) \\
&= \frac{1}{32\pi k} \omega^2 \left(a_+ a_+^* + a_\times a_\times^* \right)
\end{aligned}
$$

2.3.4 Example, with Dimensionless Units

It is convenient to use Planck units for which the constants $c, G, h/2\pi$ are all equal to one. All units are then dimensionless numbers. In particular, for the currently accepted values of the constants,

$$\begin{aligned} 1\,\mathrm{m} &= 10^{34.791}\text{Planck length units} \\ 1\,\mathrm{s} &= 10^{43.268}\text{Planck time units} \\ 1\,\mathrm{W} &= 10^{-52.560}\text{Planck power units} \end{aligned}$$

Now consider a plausible gravitational wave from an astrophysical source with a strain amplitude of about 10^{-19} and an angular frequency of $10^3\,\mathrm{rad/\,s}$ ($\frac{10^3}{2\pi} = 159.\,15\mathrm{Hz}$). In Planck units,

$$\omega = 10^3/\,\mathrm{s} = \frac{10^3}{10^{43.268}} = 5.4 \times 10^{-41}$$

Now calculate the power flux. In Planck power units per square Planck area,

$$\left\langle T^{03}_{\text{eff}} \right\rangle = \frac{1}{32\pi}\omega^2 a a^* = \frac{1}{32\pi}\left(5.4 \times 10^{-41}\right)^2 \left(10^{-19}\right)^2$$

or

$$\left\langle T^{03}_{\text{eff}} \right\rangle = 2.9 \times 10^{-121}$$

Next, figure out what a Planck power unit is.

$$1\text{Planck power unit} = \frac{1\,\mathrm{W}}{10^{-52.560}} = 3.6 \times 10^{52}\,\mathrm{W}$$

Similarly,

$$1\text{Planck length} = \frac{1\,\mathrm{m}}{10^{34.791}}$$

so

$$1\left(\text{Planck length}\right)^2 = \frac{1\,\mathrm{m}^2}{10^{34.791} \times 10^{34.791}} = 2.62 \times 10^{-70}\,\mathrm{m}^2$$

A Planck power unit per square Planck length turns out to be a very large unit of flux, namely

$$1\text{Plank flux unit} = \frac{3.6 \times 10^{52}\,\mathrm{W}}{2.62 \times 10^{-70}\,\mathrm{m}^2} = 1.37 \times 10^{122}\frac{\mathrm{W}}{\mathrm{m}^2}$$

The power flux in our example is therefore

$$\begin{aligned} \left\langle T^{03}_{\text{eff}} \right\rangle &= 2.9 \times 10^{-121}\text{Plank flux units} \\ &= 2.9 \times 10^{-121} \times 1.37 \times 10^{122}\frac{\mathrm{W}}{\mathrm{m}^2} \\ &= 40\frac{\mathrm{W}}{\mathrm{m}^2} \end{aligned}$$

Several lessons can be drawn from this example. First, a very small fluctuation in the geometry of spacetime — one part in 10^{19} — corresponds to a quite respectable power flux. If it were light, you could read by it. Second, one can figure out the power radiated by the source of such a signal by multiplying the flux by the area of a sphere.

$$\text{Source Power} = 4\pi r^2 \times 40 \frac{\text{W}}{\text{m}^2}$$

If we take $r = 150,000$ light years (the distance to the 1987a supernova), or $1.5 \times 10^5 \times 3 \times 10^8 \times 3.15 \times 10^7\ \text{m} = 1.42 \times 10^{21}\ \text{m}$ then the source power works out to

$$\text{Source Power} = 4\pi \left(1.42 \times 10^{21}\ \text{m}\right)^2 \times 40 \frac{\text{W}}{\text{m}^2} = 1.0 \times 10^{45}\ \text{W}.$$

The peak power output of a typical supernova explosion is about 10^{44} W so this number is a bit high but in the right ballpark.

2.4 The Quadrupole Radiation Formula

2.4.1 Formal Solution

Now consider a source of gravitational radiation that produces a varying stress-energy tensor. The resulting gravitational waves are found by solving the equation

$$-\Box \bar{h}^{\mu\nu} = 16\pi k T^{\mu\nu}$$

Recall that this equation used the gauge conditions

$$\partial_\nu \bar{h}^{\mu\nu} = 0$$

but not the additional conditions of the radiation gauge. The formal solution of such a wave equation is just

$$\bar{h}^{\mu\nu}\left(x^0, x^i\right) = 4 \int \frac{[T^{\mu\nu}]_{\text{ret}}}{|x - x'|} d^3x'$$

where

$$[T^{\mu\nu}]_{\text{ret}}\left(x^0, x^{j'}\right) = T^{\mu\nu}\left(x^0 - |x - x'|, x^{j\prime}\right)$$

and

$$|x - x'| = \sqrt{\delta_{ij}\left(x^i - x^{i'}\right)\left(x^j - x^{j'}\right)}$$

2.4.2 Small Source

A key assumption is that the source is small in comparison to the distance to the field point, so that the function $|x - x'|$ is nearly constant where $T^{\mu\nu}$ is not zero and we can make the replacement

$$|x - x'| \to r = \sqrt{\delta_{ij} x^i x^j}$$

and obtain

$$\bar{h}^{\mu\nu}\left(x^0, x^i\right) = \frac{4}{r}\int T^{\mu\nu}\left(x^0 - r, x^{j\prime}\right) d^3x' = \frac{4}{r}\left[\int T^{\mu\nu} d^3x\right]_{\text{ret}}$$

The key technical result we will need follows from the conservation law used twice

$$\partial_0\partial_0 T^{00} = -\partial_0\partial_\ell T^{0\ell} = -\partial_\ell\partial_0 T^{\ell 0} = \partial_m\partial_\ell T^{\ell m}$$

which gives

$$\begin{aligned}\partial_0\partial_0\left[T^{00}x^j x^k\right] &= \left[\partial_m\partial_\ell T^{\ell m}\right] x^j x^k \\ &= \partial_m\partial_\ell\left[T^{\ell m}x^j x^k\right] - 2\partial_\ell\left[T^{\ell j}x^k + T^{\ell k}x^j\right] + 2T^{jk}\end{aligned}$$

When this expression is integrated over space, the total divergence terms become surface integrals that are zero, leaving

$$\partial_0\partial_0\int T^{00}x^j x^k d^3x = 2\int T^{jk} d^3x$$

But the quadrupole moments of the source are

$$Q^{jk} = \int T^{00}x^j x^k d^3x$$

so the wave field far from the source is

$$\bar{h}^{jk}\left(x^0, x^i\right) = \frac{2}{r}\ddot{Q}^{jk}\left(x^0 - r\right)$$

2.4.3 Transverse Traceless Part

At a particular field point, x^i it is possible to remove the trace and the components in the propagation direction of the waves by making a coordinate gauge transformation. This procedure works because the coordinate variations ξ^μ obey the wave equation and can thus remove these components everywhere. Removing the trace yields

$$h^{jk}\left(x^0, x^i\right) = \frac{2}{r}\ddot{I}^{jk}\left(x^0 - r\right)$$

where

$$I^{jk} = \int T^{00}\left(x^j x^k - \frac{1}{3}\delta^{jk}r^2\right) d^3x$$

Removing the component in the propagation direction simply projects the tensor into the plane perpendicular to the radial direction.

$$h^{jk}\left(x^0, x^i\right) = \frac{2}{r}P^j{}_r P^k{}_s \ddot{I}^{rs}\left(x^0 - r\right)$$

where

$$P^j{}_r = \delta^j_r - \frac{x^j x^r}{r^2}$$

This expression gives the characteristic quadrupole radiation pattern.

2.4.4 Gravitational Wave Luminosity

To find out how much power is being emitted, note that the power flux in these units is just

$$\begin{aligned} P &= \frac{1}{32\pi}\left\langle T^{03}_{\text{eff}}\right\rangle = \frac{1}{32\pi}\left\langle \left(\frac{\partial}{\partial x^0}h^{jk}\right)\left(\frac{\partial}{\partial r}h^{jk}\right)\right\rangle = \frac{1}{32\pi}\left\langle \dot{h}^{jk}\dot{h}^{jk}\right\rangle \\ &= \frac{1}{8\pi r^2}P^j{}_r P^k{}_s \dddot{I}^{rs} P^j{}_m P^k{}_n \dddot{I}^{mn} = \frac{1}{8\pi r^2}P_{rm}P_{sn}\dddot{I}^{rs}\dddot{I}^{mn} \end{aligned}$$

or

$$P = \frac{1}{8\pi r^2}\left(\delta_{rm} - \frac{x^r x^m}{r^2}\right)\left(\delta_{rm} - \frac{x^s x^n}{r^2}\right)\dddot{I}^{rs}\dddot{I}^{mn}$$

Integrating this expression over the sphere at constant r yields the gravitational wave "luminosity" of the source.

$$L = \frac{1}{2}\dddot{I}^{rs}\dddot{I}^{mn}\frac{1}{4\pi}\int\left(\delta_{rm} - \frac{x^r x^m}{r^2}\right)\left(\delta_{rm} - \frac{x^s x^n}{r^2}\right)d\Omega$$

Now we have to do several integrals over the sphere. The resulting formula is

$$L = \frac{1}{5}\dddot{I}^{jk}\dddot{I}^{jk}$$

where repeated indexes are summed.

2.4.5 The Rotating Rod

Now consider a rod that is rotating in the $x-y$ plane. The rod starts out oriented along the x-axis in the interval

$$-\frac{d}{2} \le x \le \frac{d}{2}$$

and has cross-sectional area a. Assume that $d \gg \sqrt{a}$ so that the dominant quadrupole moment in coordinates rotating with the rod is

$$Q^{11} = a\rho\int_{-\frac{d}{2}}^{\frac{d}{2}} x^2 dx = \frac{a\rho}{12}d^3$$

and we can set the other components to zero. When the rod rotates through an angle θ, a coordinate rotation in the opposite direction will give us the non-rotating coordinates.

$$\begin{aligned} x' &= x\cos\theta - y\sin\theta \\ y' &= y\cos\theta + x\sin\theta \end{aligned}$$

And the components of the tensor Q^{ij} transform in the usual way

$$Q^{i'j'} = \frac{\partial x^{i'}}{\partial x^r}Q^{rs}\frac{\partial x^{j'}}{\partial x^s}$$

$$
\begin{aligned}
Q^{1'1'} &= \frac{\partial x^{1'}}{\partial x^1} Q^{11} \frac{\partial x^{1'}}{\partial x^1} = Q^{11}\cos^2\theta \\
Q^{1'2'} &= \frac{\partial x^{1'}}{\partial x^1} Q^{11} \frac{\partial x^{2'}}{\partial x^1} = Q^{11}\cos\theta\sin\theta \\
Q^{2'2'} &= \frac{\partial x^{2'}}{\partial x^1} Q^{11} \frac{\partial x^{2'}}{\partial x^1} = Q^{11}\sin^2\theta
\end{aligned}
$$

Because the trace term is not changing, there is no need to calculate it. The term has no effect on the time derivatives, so

$$
\dddot{I}^{j'k'} = \dddot{Q}^{j'k'}
$$

and

$$
\begin{bmatrix} \dddot{I}^{1'1'} & \dddot{I}^{1'2'} \\ \dddot{I}^{2'1'} & \dddot{I}^{2'2'} \end{bmatrix} = \frac{a\rho}{12} d^3 \frac{d^3}{dt^3} \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}
$$

Assume a constant rotation rate so that

$$
\dot{\theta} = \omega
$$

and

$$
\ddot{\theta} = 0.
$$

$$
\begin{aligned}
\frac{d^3}{dt^3} \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} &= \omega^3 \frac{d^3}{d\theta^3} \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \\
&= \omega^3 \begin{bmatrix} 8\cos\theta\sin\theta & -4\cos^2\theta + 4\sin^2\theta \\ -4\cos^2\theta + 4\sin^2\theta & -8\cos\theta\sin\theta \end{bmatrix} \\
&= 4\omega^3 \begin{bmatrix} 2\cos\theta\sin\theta & -\cos^2\theta + \sin^2\theta \\ -\cos^2\theta + \sin^2\theta & -2\cos\theta\sin\theta \end{bmatrix} \\
&= 4\omega^3 \begin{bmatrix} \sin 2\theta & -\cos 2\theta \\ -\cos 2\theta & -\sin 2\theta \end{bmatrix}
\end{aligned}
$$

The moments are then

$$
\begin{bmatrix} \dddot{I}^{1'1'} & \dddot{I}^{1'2'} \\ \dddot{I}^{2'1'} & \dddot{I}^{2'2'} \end{bmatrix} = \frac{a\rho}{3} d^3 \omega^3 \begin{bmatrix} \sin 2\theta & -\cos 2\theta \\ -\cos 2\theta & -\sin 2\theta \end{bmatrix}
$$

The luminosity of this object is then

$$
\begin{aligned}
L &= \frac{1}{5} \mathrm{Tr} \begin{bmatrix} \dddot{I}^{1'1'} & \dddot{I}^{1'2'} \\ \dddot{I}^{2'1'} & \dddot{I}^{2'2'} \end{bmatrix}^2 \\
&= \frac{1}{5} \frac{a^2\rho^2}{9} d^6 \omega^6 \mathrm{Tr} \begin{bmatrix} \sin 2\theta & -\cos 2\theta \\ -\cos 2\theta & -\sin 2\theta \end{bmatrix}^2
\end{aligned}
$$

Note that

$$\begin{aligned}\left[\begin{array}{cc}\sin 2\theta & -\cos 2\theta \\ -\cos 2\theta & -\sin 2\theta\end{array}\right]^2 &= \left[\begin{array}{cc}\cos^2 2\theta+\sin^2 2\theta & 0 \\ 0 & \cos^2 2\theta+\sin^2 2\theta\end{array}\right] \\ &= \left[\begin{array}{cc}1 & 0 \\ 0 & 1\end{array}\right]\end{aligned}$$

so we finally get

$$L=\frac{2}{45}a^2\rho^2 d^6\omega^6$$

One way to express this result is to notice that the tip speed of the rotor is just

$$v=\frac{d}{2}\omega$$

so that

$$d^6\omega^6=(2v)^6=64v^6$$

and the mass per unit length of the rotor is

$$\lambda=a\rho$$

and the luminosity takes the form

$$L=\frac{128}{45}\lambda^2 v^6=\frac{128}{45}\lambda^2\left(\frac{v}{c}\right)^6$$

Now do a bit of purely classical physics and work out the tension at the center of the rod. A segment of length Δx has mass $\Delta m=\lambda\Delta x$ and requires an unbalanced force

$$\Delta F=-\Delta m\omega^2 x=-\omega^2\lambda x\Delta x$$

to keep it on its circular path. Integrate

$$dF=-\omega^2\lambda x dx$$

to obtain

$$F(x)=F_0-\frac{1}{2}\omega^2\lambda x^2$$

The tension is zero at the ends of the rod, so

$$F(d/2)=0=F_0-\frac{1}{2}\omega^2\lambda\left(\frac{d}{2}\right)^2$$

and the central tension is then

$$F_0=\frac{1}{2}\lambda v^2$$

The breaking strength S of a material is usually expressed as a force per unit cross sectional area, so

$$F_0=Sa$$

and thus

$$Sa = \frac{1}{2}\rho a v^2$$

or

$$2S = \rho v^2$$

$$v^2 = \frac{2S}{\rho}$$

and the luminosity becomes

$$L = \frac{128}{45}\lambda^2 \left(\frac{2S}{\rho c^2}\right)^3$$

2.4.6 Playing with Rotating Rods

The simplest sort of rotating rod would be some type of fundamental string. That would rotate with a tip speed equal to c so that the luminosity of such a thing would be

$$L = \frac{128}{45}\lambda^2$$

where λ is the mass per unit length of the string, in Planck units. If such a string started out with λ of order 1 then this formula suggests that it would radiate with a Planck luminosity near 1 or about 10^{52} W and would lose its mass energy very quickly. Of course, the approximations that went into this formula do not apply here but the basis result, that fundamental strings couple very strongly to gravitational waves does appear to be valid.

For ordinary materials, the figure of merit is the dimensionless strength to mass density ratio

$$\beta^2 = \frac{2S}{\rho c^2}$$

Steel	$S = 0.7 \times 10^9$ Pa	$\rho = 7800\,\mathrm{kg/m^3}$	2.0×10^{-12}
Kevlar 149	$S = 3.4 \times 10^9$ Pa	$\rho = 1470\,\mathrm{kg/m^3}$	5.1×10^{-11}
Nanotube	$S = 63 \times 10^9$ Pa	$\rho = 2200\,\mathrm{kg/m^3}$	6.4×10^{-10}

For a macroscopic rod, put in a length density near 100 kg/m^3. The dimensionless unit of length density is one Planck Mass (2.177×10^{-8}kg) per Planck length unit (1.616×10^{-35}m) or

$$\text{Planck length density} = \frac{2.177 \times 10^{-8}}{1.616 \times 10^{-35}}\frac{\mathrm{kg}}{\mathrm{m}} = 1.35 \times 10^{27}\frac{\mathrm{kg}}{\mathrm{m}}$$

so we could assume

$$\lambda = 10^{-25}$$

which would yield the luminosity

$$L = \frac{128}{45} \times 10^{-50}\left(\beta^2\right)^3$$

Since the Planck power unit is 3.6×10^{52} W the result is

$$L = \frac{128}{45} \times 3.6 \times 10^{2} \left(\beta^{2}\right)^{3} \text{ W}$$

$$L = 1024 \left(\beta^{2}\right)^{3} \text{ W}$$

Putting in the values for different materials,

$$\begin{aligned}
L_{\text{steel}} &= 1024 \left(2.0 \times 10^{-12}\right)^{3} \text{ W} = 8.2 \times 10^{-33} \text{ W} \\
L_{\text{kevlar}} &= 1024 \left(5.1 \times 10^{-11}\right)^{3} \text{ W} = 1.36 \times 10^{-28} \text{ W} \\
L_{\text{nanotube}} &= 1024 \left(6.4 \times 10^{-10}\right)^{3} \text{ W} = 2.7 \times 10^{-25} \text{ W}
\end{aligned}$$

2.4.7 Room at the Bottom

So far, we are supposing a huge rotor perhaps five or ten meters long. However a set of smaller rotors, all rotating together would work somewhat better and the analysis would be the same. For N rotors, each with mass m and length d the luminosity would be

$$L = \frac{128}{45} \left(\frac{Nm}{d}\right)^{2} \left(\frac{2S}{\rho c^{2}}\right)^{3}.$$

This formula assumes that all of the rotors are within a small fraction of a wavelength of each other. In that case, the metric variations that they generate simply add together. Assume that the rotors all have shapes characterized by

$$a = \frac{1}{b} d^{2}$$

so that

$$m = ad\rho = \frac{1}{b} d^{3} \rho$$

and the total mass of the system is then

$$M = Nm = N \frac{1}{b} d^{3} \rho$$

and

$$\begin{aligned}
d^{3} &= b \frac{M}{N\rho} \\
d &= \left(b \frac{M}{N\rho}\right)^{1/3} = N^{-1/3} \left(M \frac{b}{\rho}\right)^{1/3} \\
\frac{Nm}{d} &= \frac{M}{\left(b \frac{M}{N\rho}\right)^{1/3}} = N^{1/3} M^{2/3} \left(\frac{\rho}{b}\right)^{1/3}
\end{aligned}$$

and the luminosity of the combination is

$$L = \frac{128}{45} N^{2/3} M^{4/3} \left(\frac{\rho}{b}\right)^{2/3} \left(\frac{2S}{\rho c^2}\right)^3$$

Dividing the total mass, M into N smaller rotors multiplies the luminosity by $N^{2/3}$ while dividing the length of each rotor by $N^{1/3}$. If we take $N = 10^{24}$ then the rotors would be of molecular scale and the formula would multiply the luminosity by 10^{16}. If the rotors have properties similar to carbon nanotubes, then a naive use of the formula would suggest a luminosity of about 10^{-9} W.

To do even better, note that most of the mass of a material is in its nuclei. If those have quadrupole moments and can be persuaded to rotate together, then the effective rotor length becomes smaller by a factor of 10^4. That alone multiplies the luminosity by another factor of 10^8 and gets us to about a tenth of a watt. The strength to mass ratio of nuclei is also much higher than chemically bound materials, so a nuclear rotor system would be capable of producing large amounts of gravitational wave power.

There is a very big catch to this method of increasing the luminosity of gravitational wave generators. The gravitational interaction is smaller than the electromagnetic interaction by about 20 orders of magnitude. The energy stored in a system of molecular scale rotors would couple to the electromagnetic field much more easily than it does to gravity and the energy would dissipate electromagnetically before any gravitational waves could be generated. The collection of rotating nanotubes that we have been contemplating would actually release all of its energy as a high energy pulse of infrared radiation that would immediately be converted into heat. Since the energy stored in the contemplated ton or so of spinning buckytubes would be more than is normally stored in chemical explosives, the result would be impressive. The nuclear rotor system would, of course, produce an even more impressive explosion.