HIGHER RANK SERIES AND ROOT PUZZLES FOR PLUMBED 3-MANIFOLDS

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Abstract. For a triple consisting of a weakly negative definite plumbed 3-manifold, a root lattice, and a generalized Spin^c-structure, we construct a family of invariants in the form of a Laurent series. Each series is an invariant of the triple up to orientation preserving homeomorphisms and the action of the Weyl group. We show that there are infinitely many such series for irreducible root lattices of rank at least 2, with each series depending on a solution to a combinatorial puzzle defined on the root lattice. Our series recover certain related series recently defined by Gukov-Pei-Putrov-Vafa, Gukov-Manolescu, Park and Ri as special cases. Explicit computations are given for Brieskorn homology spheres, for which the series may be expressed as modified higher rank false theta functions.

Introduction

The Witten-Reshetikhin-Turaev (WRT) invariants provide a powerful framework for constructing invariants for links and 3-manifolds. The construction uses as input the data of a modular tensor category and the choice of a presentation of the 3-manifold via Dehn surgery on a framed link in the 3-sphere, with the resulting invariants being independent of this choice.

An ongoing pursuit in quantum topology revolves around the categorification of WRT invariants. Recent progress has been made in this direction, particularly through a physical definition of new invariant series for 3-manifolds in Gukov-Pei-Putrov-Vafa [GPPV] and Gukov-Manolescu [GM]. These series, usually denoted as $\widehat{Z}(q)$, require the choice of a Spin^c-structure on the 3-manifolds. They arise as the Euler characteristic of a physically-defined homology theory [GPV, GPPV] and are expected to converge to the WRT invariants in some appropriate limits. A mathematical definition of such invariant series is currently available only for the 3-manifolds known as *weakly negative definite* plumbings, a class that contains for example all negative definite plumbings and is reviewed in §1.4. For this class, we extend the construction of these series and obtain infinitely many new invariant series by incorporating the data of an irreducible root lattice of rank at least 2.

Our quest was initially inspired by Akhmechet-Johnson-Krushkal [AJK], who show that in the case of *negative definite* plumbed 3-manifolds, the series $\widehat{Z}(q)$ fits in an infinite family of invariant series. Specifically, an invariant series for a negative definite plumbed 3-manifold together with a Spin^c-structure is defined starting from a plumbing presentation for the 3-manifold. As any two plumbing presentations are related via a series of two Neumann moves (§1.5), proving invariance is equivalent to checking

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invariance under the two Neumann moves. Moreover, the series $\widehat{Z}(q)$ is invariant with respect to conjugation of Spin^c-structures. For an arbitrary series, the invariance under the two Neumann moves and under the conjugation of Spin^c-structures imposes some constraints on the series coefficients. It is shown in [AJK] that there are infinitely many solutions to such constraints, with the series $\widehat{Z}(q)$ giving one example. Explicit computations in the case of Brieskorn spheres were presented in [LM].

When considering the more general case of *weakly* negative definite plumbed 3-manifolds, any two plumbing presentations are related via a series of five Neumann moves, with three extra moves complementing the two moves from the negative definite case. While the series $\widehat{Z}(q)$ (specifically, its refinement from Ri [Ri]) remains invariant, it is natural to ask whether $\widehat{Z}(q)$ is unique in this regard. In other words, we ask whether there are modifications of the series $\widehat{Z}(q)$ which satisfy the constraints given by the five Neumann moves and the conjugation of Spin^c-structures and thus remain invariant for weakly negative definite plumbings.

Before addressing this question — the answer will be given in the next Corollary 1 — we point out that Park [Par] defines more generally an invariant series $\widehat{Z}(q)$ for weakly negative definite plumbed 3-manifolds starting from the data of an arbitrary *root lattice*, with the series from [GPPV, GM] coinciding with the data of the root lattice A_1. The series requires the choice of a *generalized* Spin^c-structure depending on the root lattice, reviewed in §1.6. The space of generalized Spin^c-structures is affinely isomorphic to the first homology group of the manifold with coefficients in the root lattice (1.4). The Weyl group of the root lattice acts on the generalized Spin^c-structures, extending the conjugation of the Spin^c-structures in the A_1 case. The series $\widehat{Z}(q)$ from [Par] is invariant under this action of the Weyl group.

Thus we ask more generally whether, for an arbitrary root lattice, the invariance under the five Neumann moves and under the action of the Weyl group uniquely determines the series $\widehat{Z}(q)$ from [Par]. Our main result provides a general construction that yields the single series $\widehat{Z}(q)$ for the root lattice A_1 and infinitely many invariant series for irreducible root lattices of rank at least 2.

Specifically, let M be a weakly negative definite plumbed 3-manifold, and let Q be a root lattice. We construct a series $\mathsf{Y}_{P,a}(q)$ depending on the ancillary data consisting of a choice of a generalized Spin^c-structure a and what we call an *admissible series* $P(z)$. We define these admissible series in §2 as the solutions of a puzzle on the root lattice Q, generalizing the generating series of the *Kostant partition function*. We prove:

Theorem 1. *The series* $\mathsf{Y}_{P,a}(q)$ *is:*

(i) invariant up to orientation preserving homeomorphisms of M *and*

(ii) invariant under the action of the Weyl group W *on* a*, i.e.,*

$$\mathsf{Y}_{P,a}(q) = \mathsf{Y}_{P,w(a)}(q), \quad \text{for } w \in W.$$

When the admissible series $P(z)$ is obtained from the generating series of the Kostant partition function of Q as in (2.5), the series $\mathsf{Y}_{P,a}(q)$ recovers the series $\widehat{Z}(q)$ from [GPPV, GM] for $Q = A_1$ and the series $\widehat{Z}(q)$ from [Par] for arbitrary Q. A study of admissible series $P(z)$ shows:

Theorem 2. *(i) There exist only two admissible series $P(z)$ for the root lattice A_1, conjugate under the action of the Weyl group $\mathbb{S}_2$.*

(ii) There exist infinitely many admissible series $P(z)$ for an irreducible root lattice of rank at least 2.

Since $\mathsf{Y}_{P,a}(q)$ depends on $P(z)$ only up to the action of the Weyl group (see Remark 3.2(i)), we deduce:

Corollary 1. *For a weakly negative definite plumbed 3-manifold M, a root lattice Q, and a generalized Spin^c-structure a:*

(i) There exists a unique series $\mathsf{Y}_{P,a}(q)$ for $Q = A_1$ coinciding with $\widehat{Z}(q)$.

(ii) There exist infinitely many series $\mathsf{Y}_{P,a}(q)$ if Q is irreducible and of rank at least 2.

Brieskorn spheres and higher rank false theta functions. As an example, consider a Brieskorn homology sphere $M = \Sigma(b_1, b_2, b_3)$. In this case, there exists only one generalized Spin^c-structure, hence we remove a from the notation of the series.

Corollary 2. *Let $M = \Sigma(b_1, b_2, b_3)$ be a Brieskorn sphere with pairwise coprime integers $2 \le b_1 < b_2 < b_3$. For a root lattice Q and an admissible series*

$$P(z) = \sum_{\alpha \in 2\rho + 2Q} c(\alpha) z^{\alpha},$$

where ρ is the Weyl vector, one has

$$\mathsf{Y}_P(q) = q^C \sum_{w_1, w_2 \in W} (-1)^{\ell(w_1 w_2)} \Psi_P\left(b_1 b_2 b_3, \eta_{w_1, w_2}\right)$$

where C is a rational number (given in (6.5)),

$$\Psi_P(d, \eta) := \sum_{\alpha \in -2\rho + 2Q} c(\alpha) \sum_{w \in W} (-1)^{\ell(w)} q^{\frac{1}{8d}||d\alpha + w(\eta)||^2},$$

and

$$\eta_{w_1, w_2} := 2b_2 b_3 w_1(\rho) + 2b_1 b_3 w_2(\rho) + 2b_1 b_2 \rho \in Q \quad \text{for } w_1, w_2 \in W. \tag{0.1}$$

For $Q = A_1$, it was observed in [GM] and in earlier computations of Lawrence-Zagier [LZ] and Chung [Chu] that the series $\widehat{Z}(q)$ for a Brieskorn homology sphere is a sum of false theta functions. This is reflected in the formula in Corollary 2, since for $Q = A_1$, one has that α is odd and $c(\alpha) = 1$ if α is negative, while $c(\alpha) = 0$ otherwise. Thus in this case $\Psi_P(d, \eta)$ equals an Eichler integral of a weight $3/2$ theta function. For the Poincaré homology sphere $\Sigma(2, 3, 5)$, this holds up to an additional summand, see [GM]. For

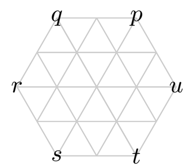

FIGURE 1. An hexagon in A_2. Here the labels p, q, r, s, t, u represent the values of $c(\alpha)$ at the corresponding $\alpha \in A_2$.

Seifert fibered spaces with three singular fibers, the radial limits of the false theta function at roots of unity agree with the WRT invariants [LZ, Thm 3]; hence the conjectural convergence of $\widehat{Z}(q)$ to the WRT invariants is verified for the Poincaré homology sphere, as observed in [GM].

Similarly, for a higher rank Q, it was observed in [Par] that the series $\widehat{Z}(q)$ for a Brieskorn homology sphere is a sum of higher rank false theta functions. Again, this is reflected in Corollary 2, as for $c(\alpha)$ being determined by the Kostant partition function as in (2.4), the series $\Psi_P(d, \eta)$ is a higher rank false theta function. Thus Corollary 2 shows how the series $\mathsf{Y}_P(q)$ for a Brieskorn homology sphere is a sum of *modified* higher rank false theta functions depending on the admissible series $P(z)$.

Root puzzles. Next, we anticipate the definition of admissible series in the case of the root lattice $Q = A_2$, and refer to §2 for arbitrary root lattices. In the A_2 case, an admissible series is equivalent to a function

$$c\colon 2A_2 \to R, \qquad \alpha \mapsto c(\alpha),$$

with R a commutative ring, satisfying the following two properties:

(i) For $\alpha \in Q$, one has $c(n\alpha) = 0$ for $n \in \mathbb{Z}$ and either $n \gg 0$ or $n \ll 0$;

(ii) For every hexagon in A_2 as in Figure 1, one has

$$-p + q - r + s - t + u = \begin{cases} 1 & \text{if the hexagon is centered at } 0 \in A_2, \\ 0 & \text{otherwise.} \end{cases}$$

Finding such a function c entails solving a combinatorial puzzle on the root lattice. The two properties (i)-(ii) guarantee that the resulting series $\mathsf{Y}_{P,a}(q)$ for a 3-manifold is invariant under the Neumann moves.

An example of such a function is provided by an appropriate affine transformation of the Kostant partition function, see Figure 3 and §§2.2 and 2.6. We give two ways to construct new admissible functions from a given admissible function yielding infinitely many examples in §2.7.

We define a similar puzzle for an arbitrary root lattice in §2, with the above properties (i)-(ii) generalized by the next properties (P1) and (P2), see Lemmata 2.3 and 2.5. The Kostant partition function provides again a solution, and we show how to construct infinitely many examples in §2.8, hence Corollary 1(ii) follows.

Open questions. We conclude this introduction with some open questions. As the series $\widehat{Z}(q)$ was originally constructed as the Euler characteristic of a physically-defined homology theory, it would be interesting to see whether the series $\mathsf{Y}_{P,a}(q)$ for an arbitrary admissible $P(z)$ coincides with the Euler characteristic of some homology theory as well.

Moreover, the root lattice Q in [Par] was determined by a connected, simply connected, semisimple Lie group acting as gauge group. It would be interesting to see whether a gauge group plays some role in the geometry behind the series $\mathsf{Y}_{P,a}(q)$.

For the series $\widehat{Z}(q)$, the minimal exponent of q appearing in the series is essentially the Heegaard-Floer d-invariant (see [AJK, Rmk. 3.3]). Upon varying the root lattice Q and the admissible series $P(z)$, a natural question is whether the minimal exponent of q appearing in the series $\mathsf{Y}_{P,a}(q)$ holds a similar geometric interpretation.

Finally, the series $\widehat{Z}(q)$ for a Brieskorn sphere and $Q = A_1$ is an example of a quantum modular form, see Zagier [Zag]. This fact extends to a larger class of 3-manifolds, see [BMM2, BMM1]. It would be interesting to determine whether the series $\mathsf{Y}_P(q)$ from Corollary 2 enjoy this property as well.

Structure of the paper. Root lattices, plumbings, Neumann moves, and Spin^c-structures are reviewed in §1. We define and study the admissible series $P(z)$ in §2 and prove Theorem 2 there. The series $\mathsf{Y}_{P,a}(q)$ is defined in §3. The proof of Theorem 1 is presented in §5 by applying auxiliary results from §4. Finally, the proof of Corollary 2 is presented in §6.

1. Notation and background

Here we review the necessary background on root lattices, plumbed 3-manifolds and their homology, Neumann moves, reduced plumbing trees, and generalized Spin^c-structures.

1.1. Root lattices.

We start by reviewing some basic facts on root lattices that will be used throughout; we refer to [Bou, Hum] for more details.

A *root system* is a pair (V, Δ) where V is a finite-dimensional Euclidean space over $\mathbb{R}$ with a positive definite bilinear form $\langle , \rangle$, and $\Delta \subset V$ is a finite subset of non-zero vectors, called *roots*, such that:

(i) $\mathbb{R}\Delta = V$;
(ii) for $\alpha \in \Delta$, one has $n\alpha \in \Delta$ if and only if $n = \pm 1$;
(iii) Δ is closed under the reflections through the hyperplanes orthogonal to the roots; and
(iv) for $\alpha, \beta \in \Delta$, one has $2\frac{\langle \alpha,\beta\rangle}{\langle \alpha,\alpha\rangle} \in \mathbb{Z}$.

Let Q be a *root lattice*, that is, $Q = \mathbb{Z}\Delta$ for some root system (V, Δ). We will denote its rank as $r := \operatorname{rank}(Q)$. The corresponding *weight lattice* P is defined as

$$P := \left\{ \lambda \in V \;\middle|\; 2\frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z} \text{ for } \alpha \in \Delta \right\}.$$

Select a set $\Delta^+ \subset \Delta$ of *positive roots*. This is the set of all roots lying on a fixed side of a hyperplane in V which does not contain any root. The *Weyl vector* $\rho \in P \cap \frac{1}{2}Q$ is defined as the half-sum of the positive roots.

A root $\alpha \in \Delta^+$ is *simple* if α cannot be written as the sum of two elements in Δ^+. Simple roots form a basis for V. For simple roots $\alpha_1, \ldots, \alpha_r$, the *fundamental weights* $\lambda_1, \ldots, \lambda_r$ are elements of P such that $2\frac{\langle \lambda_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \delta_{i,j}$ for $i, j = 1, \ldots, r$. These also form a basis of V.

Let W be the *Weyl group* acting on Q. This is the group generated by reflections through the hyperplanes orthogonal to the roots. For $w \in W$, the *length* $\ell(w)$ of w is defined as the minimum length of any expression of w as product of such reflections. This is also equal to the number of positive roots transformed by w into negative roots.

Root lattices are classified by Dynkin diagrams. As an example, the root lattice $Q = A_1$ is $\mathbb{Z}$ with bilinear form $\langle m, n \rangle = 2mn$ for $m, n \in \mathbb{Z}$. In this case, $\rho = \frac{1}{2}$ and $W = \mathbb{S}_2$ (the symmetric group on a set of size 2).

More generally, it will be convenient to have the following example in mind. The root lattice $Q = A_2$ is $\mathbb{Z}\alpha \oplus \mathbb{Z}\beta$ with $\langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle = 2$ and $\langle \alpha, \beta \rangle = -1$. In particular, the angle between α and β is $120°$, and Q is the vertex arrangement of the tiling of the Euclidean plane by equilateral triangles. In this case, $\Delta = \{\pm\alpha, \pm\beta, \pm(\alpha + \beta)\}$ and $W = \mathbb{S}_3$. For the set of positive roots $\Delta^+ = \{\alpha, \beta, \alpha + \beta\}$, the Weyl vector is $\rho = \alpha + \beta$.

1.2. **Plumbings.** We will consider closed oriented 3-manifolds that arise from the plumbing construction. Here we sketch the construction and set the notation; we refer to [Neu] and [Ném, §3.3] for details.

One starts from a plumbing graph Γ. This consists of a graph with some decorations: for each vertex, one has two integer numbers (called the *Euler number* and the *genus* of the vertex), and for each edge, one has a sign. We assume throughout that Γ is a *tree* and that the genus of each vertex is *zero*. Since Γ has no cycles, one can assume that the sign on all edges is $+1$ [Neu] (but it will be beneficial to remember that edge signs can change; we will return to this in §§1.5-1.6). Hence, for our plumbing trees we will only record the Euler number m_v for each vertex v.

For a plumbing tree Γ, let $V(\Gamma)$ be its vertex set and $E(\Gamma)$ its edge set. Choose an ordering of its vertices $v_1, \ldots, v_s$, with $s = |V(\Gamma)|$, and let $m_1, \ldots, m_s$ be the corresponding Euler numbers. An edge between vertices v_i and v_j will be denoted by $(i, j) \in E(\Gamma)$.

The *framing matrix* B determined by Γ is the $s \times s$ symmetric matrix

$$B := (B_{ij})_{i,j=1}^{s} \quad \text{with} \quad B_{ij} := \begin{cases} m_i & \text{if } i = j, \\ 1 & \text{if } i \neq j \text{ and } (i,j) \in E(\Gamma), \\ 0 & \text{otherwise.} \end{cases}$$

(More generally, the entries B_{ij} corresponding to the edges are defined to be equal to the edge signs.) We denote by $\sigma = \sigma(B)$ the signature of B and $\pi = \pi(B)$ the number of its positive eigenvalues. One has $\sigma = 2\pi - s$.

For the plumbing construction, one starts by assigning to each vertex v of Γ an oriented disk bundle over a real surface E_v of genus equal to the genus decoration of v (i.e., genus 0 in our case), with the Euler number of the bundle equal to m_v. Then one constructs a 4-manifold $X = X(\Gamma)$ by gluing together such bundles according to the edge set $E(\Gamma)$. Let $M = M(\Gamma)$ be the boundary of X. This is a closed oriented 3-manifold, called the *plumbed* 3-*manifold* constructed from Γ. Alternatively, M may be obtained by Dehn surgery on a framed link determined by Γ consisting of unknots corresponding to the vertices of Γ, framings given by the corresponding Euler numbers, and with two unknots forming an Hopf link whenever the corresponding vertices in Γ are joined by an edge.

1.3. **On the homology of the plumbed 3-manifold.** The 4-manifold X has the same homotopy type of the space E of the s real surfaces E_v, i.e., $H_i(X;\mathbb{Z}) \cong H_i(E;\mathbb{Z})$ for $i \geq 0$. The homology of the 3-manifold M follows from Lefshetz duality, the Universal Coefficient Theorem, and the long exact sequence of the pair (X, M).

Specifically, let

$$L := H_2(X;\mathbb{Z}) \cong H_2(E;\mathbb{Z}) \cong \mathbb{Z}^s.$$

The last isomorphism is induced from the choice of an ordering of the real surfaces E_v (or equivalently, the vertices of Γ). The natural intersection pairing of $L \cong \mathbb{Z}^s$ is given by the framing matrix B.

By Lefshetz duality and the Universal Coefficient Theorem, the dual of the lattice L is

$$L' = H^2(X;\mathbb{Z}) \cong H_2(X, M;\mathbb{Z}) \cong \mathbb{Z}^s.$$

This is generated by the transversal disks D_v to the surfaces E_v at general points. Hence, the natural map $L \to L'$ in the bases $\{E_v\}_v$ and $\{D_v\}_v$ is given by the framing matrix B.

In the following, we assume that the pairing of L is non-degenerate, i.e., $\det(B) \neq 0$. In this case, one has an inclusion of lattices $B\colon L \hookrightarrow L'$. From the long exact sequence of the pair (X, M), the boundary operator $L' \cong H_2(X, M;\mathbb{Z}) \to H_1(M;\mathbb{Z})$ yields a short exact sequence

$$L'/L \hookrightarrow H_1(M;\mathbb{Z}) \twoheadrightarrow H_1(X;\mathbb{Z}).$$

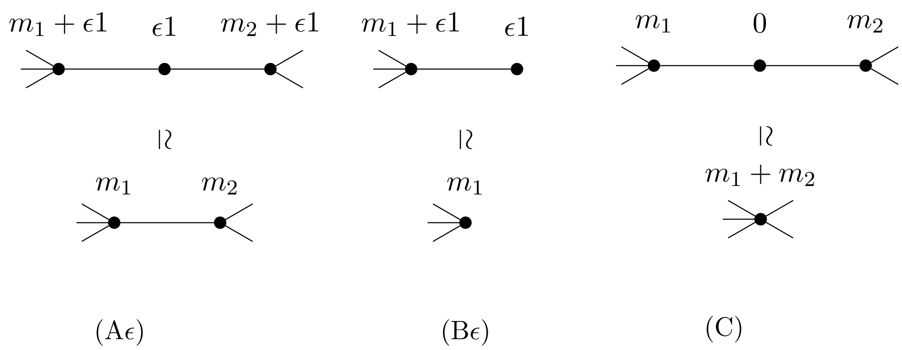

FIGURE 2. The five Neumann moves on weakly negative definite plumbing trees. Here, $\epsilon \in \{+, -\}$.

From the assumption that Γ is a tree and that all surfaces E_v have genus 0, we deduce the vanishing $H_1(X;\mathbb{Z}) \cong H_1(E;\mathbb{Z}) \cong 0$. Hence, one has

$$H_1(M;\mathbb{Z}) \cong L'/L \cong \mathbb{Z}^s/B\mathbb{Z}^s.$$

In particular, our assumptions imply that M is a rational homology sphere, i.e., $H_1(M;\mathbb{Q}) = 0$.

When $\det(B) \neq 0$, the framing matrix B is invertible over $\mathbb{Q}$, and the induced bilinear pairing on L' is

$$\langle , \rangle \colon L' \times L' \to \mathbb{Q}, \qquad (v, w) \mapsto v^t B^{-1} w.$$

This pairing is induced from B since for $x, y \in L$, one has $\langle Bx, By \rangle = x^t B y$, thus recovering the pairing of x and y in L.

For a root lattice Q and a lattice L' as above, the induced bilinear pairing on $L' \otimes_{\mathbb{Z}} Q$ is defined by

$$\langle , \rangle \colon L' \otimes_{\mathbb{Z}} Q \times L' \otimes_{\mathbb{Z}} Q \to \mathbb{Q}, \qquad (v \otimes \alpha, w \otimes \beta) \mapsto \langle v, w \rangle \langle \alpha, \beta \rangle.$$

Here the pairing $\langle v, w \rangle$ is in L' and the pairing $\langle \alpha, \beta \rangle$ is in Q. This extends by linearity as follows. For $a, b \in L' \otimes_{\mathbb{Z}} Q \cong \mathbb{Z}^s \otimes_{\mathbb{Z}} Q$, write $a = (a_1, \ldots, a_s)$ and $b = (b_1, \ldots, b_s)$, with $a_i, b_i \in Q$, for $i = 1, \ldots, s$. Then the pairing is

$$\langle a, b \rangle = \sum_{i=1}^{s} \sum_{j=1}^{s} B_{ij}^{-1} \langle a_i, b_j \rangle. \tag{1.1}$$

1.4. **Negative and weakly negative definite plumbings.** A plumbing tree Γ is *negative definite* if the framing matrix B is negative definite.

A plumbing tree Γ is *weakly negative definite* if the framing matrix B is invertible over $\mathbb{Q}$ and B^{-1} is negative definite on the subspace of $\mathbb{Z}^s$ spanned by the vertices of Γ of degree at least 3.

A plumbed 3-manifold M is *negative definite* (respectively, *weakly negative definite*) if M may be constructed from some negative definite (resp., weakly negative definite) plumbing tree, up to an orientation preserving homeomorphism.

1.5. **Neumann moves.** Here we review the Neumann moves on negative definite and weakly negative definite plumbing trees. Each move is between two plumbing trees and represents an orientation preserving homeomorphism between the corresponding plumbed 3-manifolds.

Two negative definite plumbing trees represent the same 3-manifold up to an orientation preserving homeomorphism if and only if they are related by a sequence of the moves (A−) and (B−) from Figure 2 (and their inverses) [Neu, Thm 3.2].

Two weakly negative definite plumbing trees represent the same 3-manifold up to an orientation preserving homeomorphism if and only if they are related by a sequence of the five moves in Figure 2 (and their inverses) [Neu, Thm 3.2]. However, these Neumann moves do not necessarily preserve the weakly negative definite property of the plumbing trees, see [Ri, Ex. 4.2]. In particular, a plumbing tree for a weakly negative definite plumbed 3-manifold may not necessarily be weakly negative definite, but it may become so after a sequence of the Neumann moves from Figure 2.

Remark 1.1. For each move, we will use the following observation about the framing matrices. Let Γ and $\Gamma_\circ$ be the bottom and top plumbing trees, respectively, and let B and $B_\circ$ be the corresponding framing matrices. A direct computation shows that the column space of B is isomorphic to a subspace of the column space of $B_\circ$. We show this in the case of move (A+), with the case of the other moves being similar. Assume first that Γ has only two vertices. Then B and $B_\circ$ are

$$B = \begin{pmatrix} m_1 & 1 \\ 1 & m_2 \end{pmatrix} \qquad \text{and} \qquad B_\circ = \begin{pmatrix} m_1+1 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & m_2+1 \end{pmatrix}.$$

After a column operation, $B_\circ$ becomes

$$\begin{pmatrix} m_1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & m_2 \end{pmatrix}$$

from which it is clear that the column space of B is isomorphic to a subspace of the column space of $B_\circ$. Note that the negative edge signs appearing as the two coefficients -1 of $B_\circ$ can be undone after changing the orientation of the subspace spanned by the vertex labelled by m_2. Indeed, this entails multiplying by -1 the last row and last column of $B_\circ$. For the case when Γ has more vertices, a similar block form argument applies.

1.6. **Generalized** Spin^c**-structures.** Here we review the space of generalized Spin^c-structures for a plumbed 3-manifold M and a root lattice Q. This space appeared in [Par]. A generalized Spin^c-structure will be an input of the q-series defined in §3.

Recall from §1.3 that the choice of an ordering of the vertices of Γ induces an isomorphism $L' \cong \mathbb{Z}^s$. Let

$$T_E := (2 - \deg(v_1), \ldots, 2 - \deg(v_s)) \in \mathbb{Z}^s \cong L'$$

and

$$\delta := T_E \otimes 2\rho \in L' \otimes_{\mathbb{Z}} Q \tag{1.2}$$

where ρ is the Weyl vector as in §1.1.

The *space of generalized* Spin^c*-structures* on M for the root lattice Q is

$$\mathsf{B}_Q(M) := \frac{\delta + 2L' \otimes_{\mathbb{Z}} Q}{2BL \otimes_{\mathbb{Z}} Q}. \tag{1.3}$$

For $Q = A_1$, this is simply

$$\mathrm{Spin}^c(M) \cong \frac{\delta + 2L'}{2BL},$$

the space of Spin^c-structures on M [Ném, §6.10]. Thus $\mathsf{B}_Q(M)$ generalizes the space of Spin^c-structures for an arbitrary root lattice Q. As for the case $Q = A_1$, the space $\mathsf{B}_Q(M)$ is affinely isomorphic to

$$H_1(M; Q) \cong \frac{L' \otimes_{\mathbb{Z}} Q}{BL \otimes_{\mathbb{Z}} Q}. \tag{1.4}$$

The Weyl group W naturally acts component-wise on $L' \otimes_{\mathbb{Z}} Q$, and this induces an action of W on $\mathsf{B}_Q(M)$:

$$w \colon \mathsf{B}_Q(M) \to \mathsf{B}_Q(M), \qquad [a] \mapsto [w(a)], \qquad \text{for } w \in W.$$

Proposition 1.2. *For a weakly negative definite plumbed* 3*-manifold M and a root lattice Q, the set $\mathsf{B}_Q(M)$ and the Weyl group action on it are independent of the plumbing presentation for M.*

Proposition 1.2 follows from the next Proposition 1.4. This will be used in the proof of Theorem 3.4.

To prove Proposition 1.2, it is enough to verify that $\mathsf{B}_Q(M)$ and the Weyl group action on it are invariant under the Neumann moves from Figure 2. For each move, we use the notation $B \colon L \hookrightarrow L'$ and δ defined as above for the terms related to the bottom plumbing tree Γ, and the notation $B_\circ \colon L_\circ \hookrightarrow L'_\circ$ and $\delta_\circ$ for the corresponding terms related to the top plumbing tree $\Gamma_\circ$.

For each move, we define a function

$$R \colon L' \otimes_{\mathbb{Z}} Q \to L'_\circ \otimes_{\mathbb{Z}} Q, \qquad a \mapsto R(a)$$

such that the induced map

$$\frac{\delta + 2L' \otimes_{\mathbb{Z}} Q}{2BL \otimes_{\mathbb{Z}} Q} \longrightarrow \frac{\delta_\circ + 2L'_\circ \otimes_{\mathbb{Z}} Q}{2B_\circ L_\circ \otimes_{\mathbb{Z}} Q}, \qquad [a] \mapsto [R(a)] \tag{1.5}$$

is a bijection of sets and is equivariant with respect to the action of the Weyl group W. Note that for each move, the column space of B is isomorphic to a subspace of the column space of $B_\circ$, see Remark 1.1. It follows that for each representative a of a generalized Spin^c-structure for Γ, there is a corresponding affine space of generalized Spin^c-structures for $\Gamma_\circ$, and $R(a)$ is required to be in such a space.

We proceed by defining the function R for each move. For this, we first choose an order of the vertices of Γ and a compatible order of the vertices of $\Gamma_\circ$. This induces isomorphisms $L' \cong \mathbb{Z}^s$ and $L'_\circ \cong \mathbb{Z}^{s_\circ}$, where s and $s_\circ$ are the ranks of L' and $L'_\circ$, respectively.

Remark 1.3. The choice of an order of the vertices of Γ and a compatible order of the vertices of $\Gamma_\circ$ allows one to distinguish for each Neumann move the parts of the tree that are on the left and on the right of each vertex.

Consider the Neumann move (Aϵ) from Figure 2 with $\epsilon \in \{+,-\}$. For $a \in L' \otimes_{\mathbb{Z}} Q$, write $a = (a_1, a_2)$ with subtuple a_1 corresponding to the vertices of Γ consisting of the vertex labeled by m_1 and all vertices on its left, and subtuple a_2 corresponding to the vertices of Γ consisting of the vertex labeled by m_2 and all vertices on its right. Define

$$R\colon L' \otimes_{\mathbb{Z}} Q \to L'_\circ \otimes_{\mathbb{Z}} Q, \qquad (a_1, a_2) \mapsto (a_1, 0, -\epsilon a_2) \tag{1.6}$$

with the 0 entry corresponding to the added vertex in $\Gamma_\circ$. Recall from §1.3 that all edges of plumbing graphs have a sign, which determines the corresponding gluing, and in the case of plumbing trees one can assume that all edge signs are equal [Neu]. When $\epsilon = +$, the Neumann move (A+) involves the change of an edge sign of $\Gamma_\circ$. This change of the edge sign can be undone after changing the orientation of the subspace of $L' \otimes_{\mathbb{Z}} Q$ corresponding to one side of the added vertex in $\Gamma_\circ$. Thus the minus sign multiplying a_2 in the formula for R.

Next, consider the Neumann move from Figure 2(Bϵ) with $\epsilon \in \{+,-\}$. For $a \in L' \otimes_{\mathbb{Z}} Q$, write $a = (a_\sharp, a_1)$ with entry a_1 corresponding to the vertex of Γ labeled by m_1, and subtuple $a_\sharp$ corresponding to all other vertices of Γ. Define

$$R\colon L' \otimes_{\mathbb{Z}} Q \to L'_\circ \otimes_{\mathbb{Z}} Q, \qquad (a_\sharp, a_1) \mapsto (a_\sharp, a_1 + 2\rho, \epsilon 2\rho) \tag{1.7}$$

where the entry $\epsilon 2\rho$ corresponds to the added vertex in $\Gamma_\circ$.

Finally, consider the Neumann move (C) from Figure 2. Let v_0 be the vertex in Γ labeled by $m_1 + m_2$, and let v_1, v'_0, and v_2 be the vertices in $\Gamma_\circ$ labeled by m_1, 0, and m_2, respectively. For $a \in L' \otimes_{\mathbb{Z}} Q$, write $a = (a_\sharp, a_0, a_\flat)$ with entry a_0 corresponding to the vertex v_0 in Γ, subtuple $a_\sharp$ corresponding to all vertices in Γ equivalent to the vertices of $\Gamma_\circ$ on the left of v_1, and subtuple $a_\flat$ corresponding to all vertices in Γ equivalent to the vertices of $\Gamma_\circ$ on the right of v_2. For $\beta \in Q$, define

$$R = R_\beta\colon L' \otimes_{\mathbb{Z}} Q \to L'_\circ \otimes_{\mathbb{Z}} Q, \qquad (a_\sharp, a_0, a_\flat) \mapsto (a_\sharp, a_0 + \beta, 0, \beta, -a_\flat) \tag{1.8}$$

where the entries $a_0 + \beta$, 0, and β correspond to the vertices v_1, v'_0, and v_2 in $\Gamma_\circ$, respectively. Assume that β is chosen as follows

$$\beta = \begin{cases} 2\rho & \text{if } \deg(v_1) \equiv \deg(v_2) \bmod 2, \\ 0 & \text{otherwise.} \end{cases} \tag{1.9}$$

This choice of β will allow one to verify (1.10).

Proposition 1.4. *For each Neumann move in Figure 2, the map R as defined in* (1.6)–(1.9) *induces a bijection of the sets $\mathsf{B}_Q(M)$ as in* (1.5).

The induced bijection is independent of the order of the vertices of the plumbing trees and equivariant with respect to the action of the Weyl group.

Proof. For each Neumann move in Figure 2, it is immediate to verify that R is injective. Moreover, one has

$$R\left(\delta + 2L' \otimes_{\mathbb{Z}} Q\right) \subseteq \delta_\circ + 2L'_\circ \otimes_{\mathbb{Z}} Q. \tag{1.10}$$

For instance, let us verify this for the Neumann move (C). Select

$$\ell \in \delta + 2L' \otimes_{\mathbb{Z}} Q, \qquad \text{and write} \qquad \ell = (\ell_\sharp, \ell_0, \ell_\flat).$$

This implies $\ell_0 \in (2 - \deg v_0)2\rho + 2Q$. Then in order to have

$$R_\beta(\ell) = (\ell_\sharp, \ell_0 + \beta, 0, \beta, -\ell_\flat) \in \delta_\circ + 2L'_\circ \otimes_{\mathbb{Z}} Q,$$

one needs

$$\ell_0 + \beta \in (2 - \deg v_1)2\rho + 2Q \quad \text{and} \quad \beta \in (2 - \deg v_2)2\rho + 2Q.$$

One has $\deg v_0 \equiv \deg v_1 \bmod 2$ if and only if $\deg v_1 \not\equiv \deg v_2 \bmod 2$. Hence both of these conditions are implied by the choice of β in (1.9).

The induced map (1.5) is thus well-defined and injective. Recall that the column space of B is isomorphic to a subspace of the column space of $B_\circ$, see Remark 1.1. Hence the surjectivity of the induced map (1.5) follows by a direct analysis of the extra column space of $B_\circ$. Finally, the equivariance with respect to changes of the order of the vertices of the plumbing trees and the action of the Weyl group follows immediately. □

1.7. **Reduced plumbing trees.** We will use reduced plumbing trees as in [Ri]. These are defined as follows. Let Γ be a plumbing tree. First, define a *branch* of Γ to be a path in Γ connecting a vertex of degree at least three to a vertex of degree one through a sequence of degree-2 vertices. Define a branch to be *contractible* if the branch can be contracted down to a single vertex by a sequence of the Neumann moves from Figure 2.

A vertex v of Γ is defined to be *reducible* if v has degree at least 3 but, after contracting all contractible branches incident to v, the degree of v drops down to 1 or 2.

Finally, define Γ to be *reduced* if Γ has no reducible vertices. Any plumbing tree can be reduced via a sequence of the Neumann moves from Figure 2. Note that reducing a reducible vertex to a vertex of degree 1 or 2 via a sequence of Neumann moves may yield a new reducible vertex. For this, the tree Γ becomes reduced after *repeatedly* reducing all reducible vertices via a sequence of Neumann moves. Moreover, one has:

Lemma 1.5. *By removing contractible branches, a weakly negative definite plumbing tree becomes reduced while remaining weakly negative definite.*

Proof. We argue that contracting a branch preserves the property of being weakly negative definite. A branch may be contracted by a sequence of the Neumann moves from Figure 2 of type (A±), (B±), and those moves of type (C) where at most one of the two vertices labelled by m_1 and m_2 has degree at least 3. (Moves of type (C) where the two vertices labelled by m_1 and m_2 have both degree at least 3 are not necessary to contract branches. In fact, these moves do not necessarily preserve the weakly negative definite property of the plumbing trees, see [Ri, Ex. 4.2]. Hence we avoid using them in this argument.)

For each one of these Neumann moves, let Γ and $\Gamma_\circ$ be the bottom and top plumbing graphs, respectively. We argue that if one of them is weakly negative definite, so is the other one. As in §1.6, let B and $B_\circ$ be the framing matrices of Γ and $\Gamma_\circ$, respectively, and let s and $s_\circ$ be their ranks, respectively. Let $H \subset Q^s$ and $H_\circ \subset Q^{s_\circ}$ be the subspaces spanned by the vertices of degree ≥ 3 in Γ and $\Gamma_\circ$, respectively. If the Neumann move under consideration reduces a reducible vertex to a vertex of degree 1 or 2 in Γ, then quotient $H_\circ$ by the linear subspace corresponding to that reducible vertex of degree 3 in $\Gamma_\circ$, and denote this quotient still by $H_\circ$. Thus we may assume that H and $H_\circ$ have the same rank.

We proceed to construct a map $R\colon Q^s \to Q^{s_\circ}$ which induces a linear isomorphism $H \cong H_\circ$. For the moves (A±), consider the map R from (1.6); for the moves (B±), consider the map

$$R\colon Q^s \to Q^{s+1}, \qquad (a_\sharp, a_1) \mapsto (a_\sharp, a_1, 0)$$

with notation as in (1.7); and for the move (C), consider the map $R = R_\beta$ from (1.8) with $\beta = 0$. It is immediate to see that for each move, the map R so defined induces an isomorphism $H \cong H_\circ$.

Moreover, for each $\ell \in H$, a direct computation shows that $\langle \ell, \ell \rangle = \langle R(\ell), R(\ell) \rangle$, with the pairings defined by the matrices B^{-1} and $B_\circ^{-1}$ as in (1.1), respectively. For moves (A±) and (C), this is a special case of a more general computation later done in (5.2), (5.7), (5.22); the case of moves (B±) follows similarly. Hence the statement. □

We will use a result from [Ri] showing that two reduced plumbing trees are related by a sequence of the Neumann moves from Figure 2 if and only if they are related by a sequence of those Neumann moves from Figure 2 which do not create any reducible vertices [Ri, Prop. 3.4].

2. Admissible series

Here we define and study the admissible series $P(z)$. These will be used to construct invariant series in §3. We end the section with the proof of Theorem 2.

2.1. **Admissible series.** For a root lattice Q, consider a formal series

$$P(z) := \sum_{\alpha \in 2\rho + 2Q} c(\alpha) z^{\alpha} \tag{2.1}$$

with coefficients $c(\alpha)$ in a commutative ring R. Here z^{α} for $\alpha \in Q$ (or more generally, α in the weight lattice) is a multi-index monomial defined as

$$z^{\alpha} := \prod_{i=1}^{r} z_i^{\langle \alpha, \lambda_i \rangle} \tag{2.2}$$

with $\lambda_1, \ldots, \lambda_r$ being the fundamental weights. Hence $P(z) \in R[z_1^{\pm\frac{1}{2}}, \ldots, z_r^{\pm\frac{1}{2}}]$.

Definition 2.1. A series $P(z)$ as in (2.1) is *admissible* if

$$\text{the product } P(z)P(z) \text{ is well defined} \tag{P1}$$

and

$$\left(\sum_{w \in W} (-1)^{\ell(w)} \, z^{2w(\rho)} \right) P(z) = 1. \tag{P2}$$

Before expanding on the properties (P1) and (P2), we give a key example.

2.2. **Key example.** Consider

$$W(z) := \prod_{\alpha \in \Delta^{+}} \left(\sum_{i \geq 0} z^{-(2i+1)\alpha} \right). \tag{2.3}$$

Expanding, this is

$$W(z) = \sum_{\alpha \in 2\rho + 2Q} p(\alpha) z^{\alpha}$$

with

$$p(\alpha) := \begin{array}{l} \text{number of ways to represent } -\alpha \\ \text{as a sum of odd positive multiples of positive roots.} \end{array}$$

For $Q = A_2$, the coefficients $p(\alpha)$ are represented in Figure 3.

The coefficients $p(\alpha)$ are related to the *Kostant partition function.* Specifically, for $\alpha \in Q$, the Kostant partition function is

$$k(\alpha) := \begin{array}{l} \text{number of ways to represent } \alpha \\ \text{as a sum of non-negative integer multiples of positive roots.} \end{array}$$

One has

$$p(\alpha) = \begin{cases} k\left(-\frac{\alpha}{2} - \rho\right) & \text{if } \alpha \in 2Q, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

In other words, one has

$$W(z) = z^{-2\rho} K(z^{-2}) \tag{2.5}$$

where $K(z) = \sum_{\alpha \in Q} k(\alpha) z^{\alpha}$ is the generating function of the Kostant partition function.

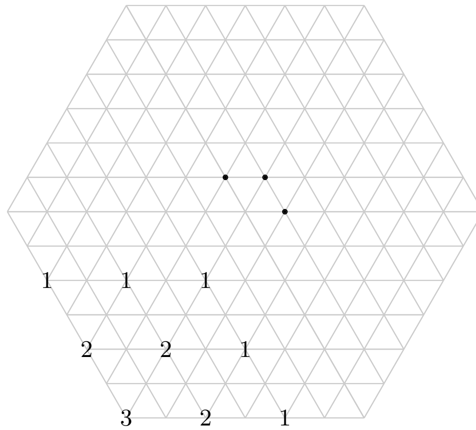

FIGURE 3. The nonzero coefficients $p(\alpha)$ for the roots of length at most 12 in the root lattice A_2. Here the three positive roots have length 2 and are marked with dots.

Applying the Weyl denominator formula

$$\sum_{w\in W} (-1)^{\ell(w)}\, z^{2w(\rho)} = \prod_{\alpha\in\Delta^+} \left(z^{\alpha} - z^{-\alpha}\right), \tag{2.6}$$

one has:

Lemma 2.2. *The series $W(z)$ is admissible.*

Proof. Property (P1) clearly holds. Moreover, property (P2) follows from the Weyl denominator formula (2.6) and the fact that

$$\left(z^{\alpha} - z^{-\alpha}\right)\left(\sum_{i\geq 0} z^{-(2i+1)\alpha}\right) = 1$$

for each α. □

2.3. **Equivalent reformulations.** Before giving more examples, it will be convenient to rephrase properties (P1) and (P2) in more explicit terms.

Lemma 2.3. *For $P(z) = \sum_{\alpha\in 2\rho+2Q} c(\alpha)z^{\alpha}$, property* (P1) *is equivalent to the condition that for $\alpha \in 2Q$, the sum*

$$\sum_{\beta\in 2\rho+2Q} c\,(\alpha+\beta)\, c\,(-\beta)$$

has only finitely many nonzero summands. Equivalently, for $\alpha \in Q$, one has $c(n\alpha) = 0$ for $n \in \mathbb{Z}$ and either $n \gg 0$ or $n \ll 0$.

Proof. Both conditions are equivalent to the fact that only finitely many terms contribute to each power of z in the expansion of the product $P(z)P(z)$. □

Before reformulating property (P2), we mention a lemma that will be used throughout. Let $\iota \in W$ be the element defined by

$$\iota(\alpha) = -\alpha \quad \text{for all } \alpha \in Q. \tag{2.7}$$

Lemma 2.4. *For $w \in W$, one has*

$$(-1)^{\ell(\iota w)} = (-1)^{|\Delta^+|}(-1)^{\ell(w)}.$$

Proof. The statement follows by comparing the coefficients of $z^{2\iota w(\rho)}$ on the two sides of the Weyl denominator formula (2.6). $\square$

Property (P2) can be rephrased as follows:

Lemma 2.5. *For $P(z) = \sum_{\alpha \in 2\rho+2Q} c(\alpha) z^\alpha$, property* (P2) *is equivalent to the condition that for $\alpha \in 2Q$, one has*

$$(-1)^{|\Delta^+|} \sum_{w\in W} (-1)^{\ell(w)}\, c\left(\alpha + 2w(\rho)\right) = \begin{cases} 1 & \text{if } \alpha = 0 \text{ in } Q, \\ 0 & \text{otherwise.} \end{cases}$$

For instance, the case $Q = A_2$ yields the rule on the hexagons from Figure 1 presented in the introduction.

Proof. Select $\alpha \in 2Q$. By Lemma 2.4, the second condition in the statement is equivalent to

$$\sum_{w\in W} (-1)^{\ell(\iota w)}\, c\left(\alpha + 2w(\rho)\right) = \begin{cases} 1 & \text{if } \alpha = 0 \text{ in } Q, \\ 0 & \text{otherwise.} \end{cases} \tag{2.8}$$

To show the equivalence of property (P2) and (2.8), write (P2) as

$$1 = \left(\sum_{w\in W} (-1)^{\ell(\iota w)}\, z^{2\iota w(\rho)} \right) P(z) = \sum_{w\in W} (-1)^{\ell(\iota w)}\, z^{2\iota w(\rho)} P(z).$$

Here the summands have been rearranged via the involution $w \mapsto \iota w$. Since for each $w \in W$, one has $w(\rho) + \iota w(\rho) = 0$, then the coefficient of z^α in

$$z^{2\iota w(\rho)} P(z)$$

is equal to the coefficient of $z^{\alpha+2w(\rho)}$ in $P(z)$. Thus the equivalence of property (P2) and (2.8). $\square$

2.4. **Weyl twists.** Here we show that twisting an admissible series by the action of elements of the Weyl group yields admissible series.

For a series $P(z) = \sum_{\alpha \in 2\rho+2Q} c(\alpha) z^\alpha$ and $w \in W$, define the *Weyl twist* of $P(z)$ by w as

$$P^w(z) := (-1)^{\ell(w)} \sum_{\alpha \in 2\rho+2Q} c(\alpha) z^{w(\alpha)}. \tag{2.9}$$

Lemma 2.6. *For an admissible series $P(z)$ and $w \in W$, the Weyl twist $P^w(z)$ is admissible.*

Proof. Property (P1) follows by applying Lemma 2.3, and property (P2) follows by applying Lemma 2.5. $\square$

2.5. **The A_1 case.** When $Q = A_1$, the series $W(z)$ from (2.3) and its Weyl twist $W^\iota(z)$ by the action of $\iota \in \mathbb{S}_2$ from (2.7) are

$$W(z) := \sum_{i\geq 0} z^{-(2i+1)} \qquad \text{and} \qquad W^\iota(z) := -\sum_{i\geq 0} z^{2i+1}.$$

Lemma 2.7. *The series $W(z)$ and $W^\iota(z)$ are the only two admissible series for the root lattice A_1.*

Proof. Assume $P(z) = \sum_{i\in\mathbb{Z}} c(i) z^i$ is admissible. For $Q = A_1$, Lemma 2.5 says that property (P2) is equivalent to

$$c\,(j-1) - c\,(j+1) = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{otherwise} \end{cases} \tag{2.10}$$

for $j \in \mathbb{Z}$. Since $P(z)$ satisfies (P1), one has $c(i) = 0$ for either $i \gg 0$ or $i \ll 0$. Assume $c(i) = 0$ for $i \gg 0$. By repeatedly applying (2.10) with $j > 0$, one has $c(i) = 0$ for $i \geq 0$. Then, since $c(0) = 0$, by repeatedly applying (2.10) with odd $j < 0$, one has $c(i) = 0$ for even $i < 0$. Also, since $c(1) = 0$, by applying (2.10) with $j = 0$, one has $c(-1) = 1$. Finally, by repeatedly applying (2.10) with even $j < 0$, one has $c(i) = 1$ for odd $i \leq -1$. Hence, $P(z) = W(z)$. Similarly, the case $c(i) = 0$ for $i \ll 0$ yields $W^\iota(z)$, hence the statement. □

2.6. **The key example in the A_2 case.** When $Q = A_2$, let α and β be the two simple roots. Then α, β and $\rho = \alpha + \beta$ are the three positive roots, and the admissible series $W(z)$ from (2.3) is

$$W(z) = \left(\sum_{i\geq 0} z^{-(2i+1)\alpha}\right)\left(\sum_{i\geq 0} z^{-(2i+1)\beta}\right)\left(\sum_{i\geq 0} z^{-(2i+1)\rho}\right).$$

Since $\rho = \alpha + \beta$, a simple computation shows that this expands as

$$W(z) = \sum_{m,n\geq 0} \min\{m,n\}\, z^{-2m\alpha - 2n\beta}. \tag{2.11}$$

Indeed, the coefficient of $z^{-2m\alpha-2n\beta}$ here follows from the computation

$$\begin{aligned}
&|\,\{(i,j,k) \in 2\mathbb{N}+1 \,|\, i\alpha + j\beta + k\rho = 2m\alpha + 2n\beta\}\,| \\
&= |\,\{(i,j,k) \in 2\mathbb{N}+1 \,|\, (i+k)\alpha + (j+k)\beta = 2m\alpha + 2n\beta\}\,| \\
&= |\,\{k \in 2\mathbb{N}+1 \,|\, k < 2m \text{ and } k < 2n\}\,| \\
&= \min\{m,n\}.
\end{aligned}$$

These values are represented in Figure 3. Equivalently, the Kostant partition function from §2.2 for A_2 is

$$k(a\alpha + b\beta) = 1 + \min\{a,b\} \quad \text{for } a,b \geq 0$$

and vanishes otherwise. Then (2.11) follows from (2.4).

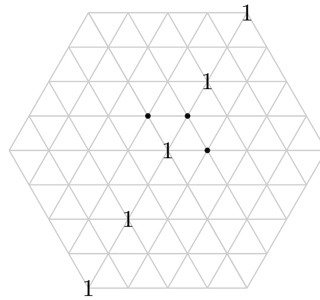

FIGURE 4. The nonzero coefficients of the even Weyl line $L_x(z)$ for $x = 1_W$ and the powers of z of length at most 8. Here the three positive roots have length 2 and are marked with dots.

2.7. **More examples in the A_2 case.** Infinitely many more examples of admissible series in the case $Q = A_2$ can be obtained as follows. Let $\sigma \in W$ be the reflection given by $\sigma(\alpha) = -\alpha$. Then σ exchanges β and ρ. Consider the Weyl twist

$$\begin{aligned} W^\sigma(z) &= -\left(\sum_{i\geq 0} z^{(2i+1)\alpha}\right)\left(\sum_{i\geq 0} z^{-(2i+1)\beta}\right)\left(\sum_{i\geq 0} z^{-(2i+1)\rho}\right) \\ &= -\sum_{m,n\geq 0} \min\{m,n\}\, z^{2m\alpha - 2n\rho}. \end{aligned}$$

Adding appropriate translates of $W(z) - W^\sigma(z)$ to $W(z)$ yields infinitely many admissible series:

Lemma 2.8. *For $\gamma \in A_2$ with $\langle \gamma, 2\rho - \alpha\rangle \geq 0$ (i.e., $\gamma \in \mathbb{Z}\alpha + \mathbb{N}_{\geq 0}\beta + \mathbb{N}_{\geq 0}\rho$), the series*

$$W(z) + z^{2\gamma}\left(W(z) - W^\sigma(z)\right)$$

is admissible.

Proof. Since both $W(z)$ and $W^\sigma(z)$ satisfy (P2), one has by linearity

$$\left(\sum_{w\in W} (-1)^{\ell(w)}\, z^{2w(\rho)}\right)\left(W(z) - W^\sigma(z)\right) = 0. \tag{2.12}$$

Then the given series satisfies (P2) by linearity. Moreover, the second reformulation of (P1) in Lemma 2.3 clearly holds due to the choice of γ, hence (P1) holds as well. □

Even more examples can be constructed as follows. For $x \in W$, consider first the *even Weyl line*

$$L_x(z) = \sum_{i\in 2\mathbb{Z}} z^{ix(\rho)}.$$

An example is in Figure 4.

Lemma 2.9. *For $Q = A_2$ and $x \in W$, one has*

$$\left(\sum_{w \in W} (-1)^{\ell(w)} \, z^{2w(\rho)} \right) L_x(z) = 0.$$

Proof. Write $L_x(z) = \sum_{\gamma \in 2Q} c(\gamma) z^\gamma$. As in the proof of Lemma 2.5, the statement is equivalent to

$$\sum_{w \in W} (-1)^{\ell(w)} \, c\,(\eta + 2w(\rho)) = 0 \tag{2.13}$$

for $\eta \in 2Q$. Expanding, the left-hand side is

$$c\,(\eta + 2\rho) - c\,(\eta + 2\beta) + c\,(\eta - 2\alpha) - c\,(\eta - 2\rho) + c\,(\eta - 2\beta) - c\,(\eta + 2\alpha)\,.$$

Since $c(\gamma) = 1$ if $\gamma \in 2x(\rho)\mathbb{Z}$ and $c(\gamma) = 0$ otherwise, at most two of these six summands are non-zero, and if two summands are non-zero, then they must have opposite coefficients. Hence (2.13) holds. □

Adding translates of $L_x(z)$ to an admissible series yields infinitely many admissible series:

Lemma 2.10. *Let $P(z)$ be an admissible series with coefficients in a commutative ring R. For elements $\gamma \neq 0$ in A_2, $c \in R$, and $x \in W$, the series*

$$P(z) + cz^{2\gamma} L_x(z)$$

is admissible.

Proof. Property (P2) follows by linearity after applying Lemma 2.9. Property (P1) follows by Lemma 2.3. □

2.8. **More examples for arbitrary Q.** The construction of Lemma 2.8 can be extended to the case of an arbitrary irreducible root lattice Q of rank at least 2 as follows. Select a simple root $\alpha \in \Delta^+$, and let $\sigma \in W$ be the reflection given by $\sigma(\alpha) = -\alpha$. Then σ permutes the positive roots other than α [Hum, Lemma B, §10.2], hence $\sigma(\rho) = \rho - \alpha$. This together with the fact that $\langle \alpha, \zeta \rangle + \langle \alpha, \sigma(\zeta) \rangle = 0$ for $\zeta \in Q$ implies

$$\langle \alpha, 2\rho - \alpha \rangle = 0.$$

In particular, $2\rho - \alpha$ is a sum of positive roots and is perpendicular to α. Adding appropriate translates of $W(z) - W^\sigma(z)$ to $W(z)$ yields infinitely many admissible series:

Lemma 2.11. *For $\gamma \in Q$ with $\langle \gamma, 2\rho - \alpha \rangle \geq 0$, the series*

$$W(z) + z^{2\gamma} \left(W(z) - W^\sigma(z) \right)$$

is admissible.

Proof. The proof of Lemma 2.8 holds verbatim. □

We can now conclude the proof of Theorem 2:

Proof of Theorem 2. Part (i) follows from Lemma 2.7 and part (ii) from Lemma 2.11. □

3. An invariant series

After defining graded Weyl twists of admissible series and Weyl assignments for reduced plumbing trees, we define a q-series and state the main theorem about its invariance. We conclude with a discussion of the relation with [GM, Par, Ri].

3.1. **Graded Weyl twists.** Let $P(z)$ be an admissible series as in §2, and for $x \in W$ recall the Weyl twist $P^x(z)$ from (2.9). For $x \in W$, define the *graded Weyl twist* of $P(z)$ as

$$P^x_n(z) := \begin{cases} \left(\sum_{w\in W} (-1)^{\ell(w)}\, z^{2w(\rho)} \right)^2 & \text{if } n = 0, \\ \sum_{w\in W} (-1)^{\ell(w)}\, z^{2w(\rho)} & \text{if } n = 1, \\ 1 & \text{if } n = 2, \\ (P^x(z))^{n-2} & \text{if } n \geq 3. \end{cases}$$

Note that $P^x_n(z)$ for $n \in \{0,1,2\}$ does not depend on $P(z)$, nor on x. Moreover, for $P(z) = \sum_{\alpha\in 2\rho+2Q} c(\alpha) z^\alpha$, expanding the case $n \geq 3$ and using the Weyl twist (2.9) yields

$$P^x_n(z) = \left((-1)^{\ell(x)} \sum_{\alpha\in 2\rho+2Q} c(\alpha) z^{x(\alpha)} \right)^{n-2} \qquad \text{if } n \geq 3. \tag{3.1}$$

The assumption (P1) implies that the $(n-2)$-power here is well defined.

Remark 3.1. From property (P2), an admissible series $P(z)$ can be interpreted as an inverse of the Laurent polynomial $\sum_{w\in W} (-1)^{\ell(w)}\, z^{2w(\rho)}$, and similarly for the Weyl twists $P^x(z)$ with $x \in W$ after Lemma 2.6. Hence, the graded Weyl twist $P^x_n(z)$ can be interpreted as an $(n-2)$-power of $P^x(z)$ also when $n \in \{0, 1, 2\}$.

3.2. **Weyl assignments.** For a reduced plumbing tree Γ (as in §1.7) with framing matrix B and a root lattice Q, a *Weyl assignment* is a map

$$\xi\colon V(\Gamma) \to W, \qquad v \mapsto \xi_v$$

such that

$$\xi_v = 1_W \qquad \text{if } \deg v \leq 2, \tag{3.2}$$

with 1_W being the identity element in W, and such that the values on vertices across what we call *forcing bridges* are coordinated by the following condition (3.3).

First, define a *bridge* of Γ to be a path in Γ connecting two vertices, both of degree at least 3, through a sequence of degree-2 vertices.

Then, define a *forcing bridge* of Γ to be a bridge of Γ that can be contracted down to a single vertex by a sequence of the Neumann moves (Aϵ)

and (C) from Figure 2. A forcing bridge of Γ between vertices v and w will be denoted by $\Gamma_{v,w}$.

Finally, for a Weyl assignment ξ, one requires

$$\xi_v = \iota^{\Delta\pi(v,w)}\xi_w \qquad \text{for every forcing bridge } \Gamma_{v,w} \tag{3.3}$$

where $\iota \in W$ is as in (2.7), and $\Delta\pi(v,w)$ is defined as the difference in numbers of positive eigenvalues

$$\Delta\pi(v,w) := \pi(B) - \pi(\overline{B}) \tag{3.4}$$

with $\overline{B}$ being the framing matrix of the plumbing tree obtained from Γ after contracting $\Gamma_{v,w}$.

Define

$$\Xi := \{\text{Weyl assignments } \xi\}.$$

One has

$$|\Xi| = |W|^n \qquad \text{where } n := |\{v \in V(\Gamma) : \deg v \geq 3\}| - |\{\text{forcing bridges}\}|.$$

3.3. **The q-series.** Let M be a weakly negative definite plumbed 3-manifold. After a sequence of Neumann moves, one can assume that M is constructed from a plumbing tree Γ which is reduced and weakly negative definite (see Lemma 1.5). For a root lattice Q, select a representative a of a generalized Spin^c-structure

$$a \in \delta + 2L' \otimes_{\mathbb{Z}} Q \subset L' \otimes_{\mathbb{Z}} Q, \tag{3.5}$$

and an admissible series $P(z)$ as in §2, with coefficients in a commutative ring R. Define

$$\mathsf{Y}_{P,a}(q) := (-1)^{|\Delta^+|\,\pi} q^{\frac{1}{2}(3\sigma - \operatorname{tr} B)\langle\rho,\rho\rangle} \sum_{\ell \in a + 2BL\otimes Q} c_\Gamma(\ell)\, q^{-\frac{1}{8}\langle\ell,\ell\rangle} \tag{3.6}$$

where

$$c_\Gamma(\ell) := \frac{1}{|\Xi|} \sum_{\xi\in\Xi} \prod_{v \in V(\Gamma)} \left[P^{\xi_v}_{\deg v}(z_v)\right]_{\ell_v}. \tag{3.7}$$

Here Ξ is the set of Weyl assignments from §3.2, P^x_n denotes a graded Weyl twist as in §3.1, the operator $[\ ]_\alpha$ assigns to a series in z the coefficient of the monomial z^α for $\alpha \in Q$, and $\ell_v \in Q$ denotes the v-component of $\ell \in L' \otimes_{\mathbb{Z}} Q \cong Q^{V(\Gamma)}$ for $v \in V(\Gamma)$. One has $c_\Gamma(\ell) \in \frac{1}{|\Xi|} R$.

Remark 3.2. (i) Replacing the admissible series $P(z)$ with a Weyl twist $P^w(z)$ for some $w \in W$, as in (2.9), yields the same series $\mathsf{Y}_{P,a}(q)$. Hence, $\mathsf{Y}_{P,a}(q)$ depends on $P(z)$ only up to Weyl twists.

(ii) When the reduced plumbing tree Γ has only vertices of degree at most 2 (i.e., Γ is a path graph), the series $\mathsf{Y}_{P,a}(q)$ does not depend on $P(z)$. This is due to the fact that the graded Weyl twists P^x_n do not depend on $P(z)$ for $n \leq 2$, see §3.1. Hence in this case, the series $\mathsf{Y}_{P,a}(q)$ coincides with the series $\widehat{Z}(q)$ from [GM, Par], see §3.5. For an example with Γ containing a vertex of degree 3, see §6.

(iii) The sum in the series is over ℓ such that $\ell \equiv a \bmod 2BL \otimes Q$. As these ℓ are all the representative of the class of a in the quotient space $\mathsf{B}_Q(M)$ from (1.3), the series $\mathsf{Y}_{P,a}(q)$ depends on a at most up to its class in $\mathsf{B}_Q(M)$. Moreover, we show in Theorem 5.1 that $\mathsf{Y}_{P,a}(q)$ depends on a only up to its class in the space $\mathsf{B}_Q^W(M)$, which is the quotient of $\mathsf{B}_Q(M)$ modulo the action of the Weyl group W.

Lemma 3.3. *Since the plumbing tree is assumed to be weakly negative definite, the powers of q in $\mathsf{Y}_{P,a}(q)$ are bounded below, and for each power of q there are only finitely many contributions to $\mathsf{Y}_{P,a}(q)$. In particular, one has*

$$\mathsf{Y}_{P,a}(q) \quad \in \quad q^{\frac{1}{2}(3\sigma - \operatorname{tr} B)\langle \rho,\rho\rangle - \frac{1}{8}\langle a,a\rangle} \frac{1}{|\Xi|} R\left(\left(q^{\frac{1}{2}}\right)\right). \tag{3.8}$$

Proof. This is similar to the argument for the series $\widehat{Z}(q)$ from [GM]. The sum over ℓ can be decomposed as a sum over the entries of ℓ corresponding to vertices of degree at most 2 and the entries corresponding to vertices of degree at least 3. For the former ones, there are only finitely many contributions due to the definition of the graded Weyl twists in §3.1. For the latter ones, the boundedness of the exponents of q and the finiteness of the contributions to each power of q follow from the assumption that the plumbing tree is weakly negative definite. Recall from §1.4 that this implies that the inverse B^{-1} of the framing matrix is negative definite on the subspace spanned by the vertices of degree at least 3.

Finally, the exponents of q in (3.8) follow from the fact that for an element $\ell \in a + 2BL \otimes Q$, one has $\langle \ell, \ell\rangle \in \langle a, a\rangle + 4\mathbb{Z}$. □

The Laurent ring in (3.8) can be simplified for $Q = A_1$: since the pairing in A_1 is always even, one has $\langle \ell, \ell\rangle \in \langle a, a\rangle + 8\mathbb{Z}$, hence (3.8) for $Q = A_1$ becomes

$$\mathsf{Y}_{P,a}(q) \quad \in \quad q^{\frac{1}{4}(3\sigma - \operatorname{tr} B - a^2)} \frac{1}{|\Xi|} R((q)).$$

Here we use that $\langle \rho, \rho\rangle = \frac{1}{2}$ for $Q = A_1$.

Our main result is:

Theorem 3.4. *Any two reduced (not necessarily weakly negative definite) plumbing trees for M yield the same series $\mathsf{Y}_{P,a}(q)$.*

This statement is the main step towards Theorem 1. We prove it in §5.

3.4. **The A_1 case and Ri's series.** Recall from §2.5 that for $Q = A_1$, there is only one admissible series $P(z)$, up to Weyl twists. The resulting series $\mathsf{Y}_{P,a}(q)$ coincides with the q-series from [Ri].

3.5. **Relation with the series $\widehat{Z}(q)$.** Assume that the reduced weakly negative definite plumbing tree Γ has no forcing bridges (e.g., this is the case when Γ has at most one vertex of degree at least 3). Then for each ℓ, the coefficient $c_\Gamma(\ell)$ from (3.7) is equal to

$$c_\Gamma(\ell) = \prod_{v\in V(\Gamma)} \left(\frac{1}{|W|} \sum_{x\in W} \left[P^x_{\deg v}(z_v)\right]_{\ell_v} \right). \tag{3.9}$$

This is due to the fact that in the absence of forcing bridges, the values of the Weyl assignments on various vertices do not need to be coordinated as in (3.3); and the fact that for $n \in \{0,1,2\}$, the graded Weyl twist $P^x_n(z)$ is independent of x, and thus

$$P^x_n(z) = \frac{1}{|W|} \sum_{x\in W} P^x_n(z) \qquad \text{for } n \in \{0,1,2\}. \tag{3.10}$$

Now, consider the case $P(z) = W(z)$ as in (2.3).

Lemma 3.5. *For arbitrary $n \geq 0$ and $\alpha \in Q$, one has*

$$\frac{1}{|W|}\left[\sum_{x\in W} W^x_n(z)\right]_\alpha$$
$$= \mathsf{v.p.}\oint_{|z_1|=1} \dots \mathsf{v.p.}\oint_{|z_r|=1} \left(\sum_{w\in W} (-1)^{\ell(w)}\, z^{2w(\rho)}\right)^{2-n} z^{-\alpha} \prod_{k=1}^{r} \frac{dz_k}{2\pi \mathrm{i} z_k}.$$

Proof. Here $\mathsf{v.p.}$ stands for the principal value (*valeur principale* in French) of the integral and is computed as follows. For $n \in \{0,1,2\}$, the term

$$\left(\sum_{w\in W} (-1)^{\ell(w)}\, z^{2w(\rho)}\right)^{2-n} \tag{3.11}$$

expands as a finite sum, and the $\mathsf{v.p.}$ integral is simply the regular integral. Hence, for $\alpha \in Q$, the right-hand side equals the coefficient of z^α in the expansion of (3.11), and the lemma holds by (3.10) and the definition of the graded Weyl twist in §3.1. (The values for $n = 1, 2$ are given in (4.2), (4.1).)

For $n \geq 3$, applying the Weyl denominator formula (2.6), one sees that the term (3.11) is singular when $z^\beta = \pm 1$ for $\beta \in \Delta^+$. In this case, the term (3.11) admits various series expansions, one in each Weyl chamber. For $\alpha \in Q$, the right-hand side is defined as the average of the coefficients of z^α among these various series expansions. As the various series expansions coincide with $W^x_n(z)$ for $x \in W$, the lemma holds. ∎

It follows that (3.9) can be expressed as a product of v.p. integrals. In particular, when the plumbing tree Γ is reduced weakly negative definite and has no forcing bridges, the series $\mathsf{Y}_{W,a}(q)$ recovers the series $\widehat{Z}(q)$ from [Par]. And for $Q = A_1$, this equals the series $\widehat{Z}(q)$ from [GPPV, GM].

4. Properties of graded Weyl twists

Here we discuss some properties of graded Weyl twists which will be used in the proof of the invariance of the q-series in §5.

Let $P(z)$ be an admissible series for a root lattice Q as in §2. For $x \in W$, consider the graded Weyl twist $P_n^x(z)$ as in §3.1 and expand as

$$P_n^x(z) = \sum_{\alpha \in Q} c_{x,n}(\alpha) z^\alpha.$$

In other words, define $c_{x,n}(\alpha)$ as the coefficient of z^α in $P_n^x(z)$. Note that one has

$$c_{x,n}(\alpha) = 0 \qquad \text{for } \alpha \notin (n-2)(2\rho + 2Q).$$

Example 4.1. For $n = 2$, one has

$$c_{x,2}(\alpha) = \begin{cases} 1 & \text{if } \alpha = 0 \text{ in } Q, \\ 0 & \text{otherwise;} \end{cases} \tag{4.1}$$

and for $n = 1$, one has

$$c_{x,1}(\alpha) = \begin{cases} (-1)^{\ell(w)} & \text{if } \alpha = 2w(\rho) \text{ in } Q \text{ for some } w \in W, \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

Recall that for $n \in \{0, 1, 2\}$, the graded Weyl twist $P_n^x(z)$ is independent of $P(z)$ and x, as confirmed in (4.1) and (4.2).

Lemma 4.2. *For $x \in W$ and $n \geq 1$, one has*

$$(-1)^{|\Delta^+|} \sum_{w \in W} (-1)^{\ell(w)} c_{x,n}(\alpha + 2w(\rho)) = c_{x,n-1}(\alpha) \qquad \textit{for } \alpha \in Q.$$

The case $n = 3$ follows from Lemma 2.5 and (4.1). The case $n = 2$ can be verified directly applying (4.1) and (4.2).

Proof. Select $x \in W$, $n \geq 1$, and $\alpha \in Q$. By Lemma 2.4, the statement is equivalent to

$$\sum_{w \in W} (-1)^{\ell(\iota w)} c_{x,n}(\alpha + 2w(\rho)) = c_{x,n-1}(\alpha) \tag{4.3}$$

where the element $\iota \in W$ is as in (2.7).

To show this, assume first $n \geq 3$. Recall that the Weyl twist $P^x(z)$ is admissible by Lemma 2.6, and rewrite property (P2) as

$$1 = \left(\sum_{w \in W} (-1)^{\ell(\iota w)} z^{2\iota w(\rho)} \right) P^x(z) = \sum_{w \in W} (-1)^{\ell(\iota w)} z^{2\iota w(\rho)} P^x(z).$$

The elements in the sums have been rearranged via the involution $w \mapsto \iota w$. Multiplying by $P_{n-1}^x(z)$ and observing that

$$P_{n-1}^x(z) P^x(z) = P_n^x(z)$$

by the definition in §3.1, one has

$$P^x_{n-1}(z) = \sum_{w\in W} (-1)^{\ell(\iota w)}\, z^{2\iota w(\rho)} P^x_n(z). \tag{4.4}$$

The coefficient $c_{x,n-1}(\alpha)$ is defined as the coefficient of z^α in $P^x_{n-1}(z)$. Hence the statement follows from (4.4) by observing that for each $w \in W$, the coefficient of z^α in

$$z^{2\iota w(\rho)} P^x_n(z)$$

is equal to the coefficient of $z^{\alpha+2w(\rho)}$ in $P^x_n(z)$ (since $w(\rho)+\iota w(\rho) = 0$ in Q).

Finally, for $n \in \{1, 2\}$, the identity (4.4) can be verified directly from the definition of graded Weyl twists in §3.1, hence the statement. □

Next, we study how the coefficients $c_{x,n}(\alpha)$ vary under Weyl reflections:

Lemma 4.3. *For $x \in W$ and $n \geq 0$, the involution $x \mapsto \iota x$ yields*

$$[P^x_n(z)]_\alpha = (-1)^{|\Delta^+|\,n} [P^{\iota x}_n(z)]_{-\alpha} \qquad \textit{for } \alpha \in Q.$$

Equivalently, one has

$$c_{x,n}(\alpha) = (-1)^{|\Delta^+|\,n} c_{\iota x,n}(-\alpha) \qquad \textit{for } \alpha \in Q.$$

Here, $\iota \in W$ is as in (2.7).

Proof. The case $n = 2$ is clear, since $P^x_2(z) = 1$. Similarly, the cases $n \in \{0, 1\}$ follow from the definition of graded Weyl twists in §3.1 and Lemma 2.4. For $n = 3$, since $P^x_3(z) = P^x(z)$, the statement follows from the definition (2.9) and Lemma 2.4. Finally, the case $n \geq 4$ follows since $P^x_n(z) = (P^x(z))^{n-2}$. □

More generally, one has:

Lemma 4.4. *For $x, w \in W$ and $n \geq 0$, one has*

$$[P^x_n(z)]_\alpha = (-1)^{\ell(w)n} [P^{wx}_n(z)]_{w(\alpha)} \qquad \textit{for } \alpha \in Q.$$

Equivalently, one has

$$c_{x,n}(\alpha) = (-1)^{\ell(w)n} c_{wx,n}(w(\alpha)) \qquad \textit{for } \alpha \in Q.$$

Proof. The statement follows by applying the same argument used for the previous lemma after replacing the use of Lemma 2.4 there with the fact that $\ell(wx) \equiv \ell(w) + \ell(x) \bmod 2$. □

Finally, one has:

Lemma 4.5. *For $x \in W$ and $p, q \geq 1$, one has*

$$P^x_p(z) P^x_q(z) = P^x_{p+q-2}(z).$$

Equivalently, one has

$$\sum_\beta c_{x,p}(\alpha+\beta)\, c_{x,q}(-\beta) = c_{x,p+q-2}(\alpha) \qquad \textit{for } \alpha \in Q.$$

Proof. The case when either $p = 1$ or $q = 1$ follows from Lemma 4.2, equation (4.2), and Lemma 2.4. The case $p, q \geq 2$ follows by the definition of the graded Weyl twists in §3.1. The second part of the statement follows by comparing the coefficients of z^α on the two sides, for $\alpha \in Q$. Note that the sum over β is finite thanks to property (P1) for the admissible series $P^x(z)$ (see Lemma 2.3). □

5. Invariance of the q-series

Here we first prove the invariance of the series $\mathsf{Y}_{P,a}(q)$ with respect to the action of the Weyl group and then prove Theorem 3.4. We conclude with the proof of Theorem 1.

Recall the coefficients $c_\Gamma(\ell)$ from (3.7).

Theorem 5.1. *For $\ell \in \delta + 2L' \otimes_{\mathbb{Z}} Q \subset L' \otimes_{\mathbb{Z}} Q$, one has*

$$c_\Gamma(\ell) = c_\Gamma(w(\ell)) \qquad \text{for } w \in W.$$

In particular, the series $\mathsf{Y}_{P,a}(q)$ is invariant by the action of the Weyl group W, that is, $\mathsf{Y}_{P,a}(q) = \mathsf{Y}_{P,w(a)}(q)$, for $w \in W$.

Proof. The first part of the statement follows after multiplying the identity in Lemma 4.4 over all vertices in the reduced plumbing tree Γ, averaging over all Weyl assignments, and applying the identity

$$\prod_{v \in V(\Gamma)} (-1)^{\deg v} = 1.$$

Indeed, one has that $\sum_{v \in V(\Gamma)} \deg v$ is even, as every edge of Γ is incident to two vertices in Γ. Moreover, the last part of the statement follows from the fact that $\langle \ell, \ell \rangle = \langle w(\ell), w(\ell) \rangle$ for $w \in W$, hence the exponent of q is also invariant by the action of W. □

We now prove Theorem 3.4:

Proof of Theorem 3.4. Let M be a weakly negative definite plumbed 3-manifold, and let Q be a root lattice. Select an admissible series $P(z)$ for Q. We verify that any two reduced plumbing trees for M yield the same series $\mathsf{Y}_{P,a}(q)$ for all representatives a of a generalized Spin^c-structure for Q. Any two reduced plumbing trees for M up to orientation preserving homeomorphisms are related by a sequence of the Neumann moves from Figure 2 which do not create any reducible vertices [Ri, Prop. 3.4].

For each such move, we argue that the two q-series arising from the two plumbing trees are equal. As in the proof of Proposition 1.2, we use the notation $B\colon L \hookrightarrow L'$ and δ for the terms related to the bottom plumbing tree Γ, and the notation $B_\circ\colon L_\circ \hookrightarrow L'_\circ$ and $\delta_\circ$ for the corresponding terms related to the top plumbing tree $\Gamma_\circ$. The signatures of B and $B_\circ$ will be denoted by σ and $\sigma_\circ$, respectively, and the numbers of positive eigenvalues of B and $B_\circ$ will be denoted by π and $\pi_\circ$, respectively.

Select a representative of a generalized Spin^c-structure a for the bottom plumbing tree. For each move, we start by observing how the factor

$$(-1)^{|\Delta^+|\,\pi} q^{\frac{1}{2}(3\sigma - \operatorname{tr} B)\langle\rho,\rho\rangle} \tag{5.1}$$

in front of the sum in the series changes under the move. Afterwards, we focus on the sum over the various representatives $\ell \in a + 2BL \otimes Q$ of the generalized Spin^c-structure. For this, recall from Proposition 1.4 that the space of generalized Spin^c-structures is invariant under the Neumann moves. However, for each move, the column space of B is isomorphic to some subspace of the column space of $B_\circ$, see Remark 1.1. It follows that for each representative ℓ of the generalized Spin^c-structure for the bottom tree, there is a corresponding affine space of generalized Spin^c-structures for the top plumbing tree. Thus for each move, we argue that the contribution of each ℓ for the bottom plumbing tree equals the sum of the contributions of the elements in the corresponding affine space for the top plumbing tree.

Step (A−): The Neumann move (A−) from Figure 2. There exists an extra term in the quadratic form corresponding to $B_\circ$ with respect to the quadratic form corresponding to B given by

$$-x_0^2 - x_1^2 - x_2^2 + 2x_0x_1 + 2x_0x_2 - 2x_1x_2 = -(x_0 - x_1 - x_2)^2,$$

where x_0 is the variable corresponding to the added vertex and x_1 and x_2 are the variables corresponding to its two adjacent vertices in $\Gamma_\circ$. It follows that

$$\sigma_\circ = \sigma - 1 \qquad \text{and} \qquad \pi_\circ = \pi.$$

Since $\operatorname{tr} B_\circ = \operatorname{tr} B - 3$, one has $3\sigma_\circ - \operatorname{tr} B_\circ = 3\sigma - \operatorname{tr} B$. We conclude that the factor (5.1) in front of the sum in the series is invariant under this move.

Next, we consider the sum in the series. Recall the function R from (1.6) with $\epsilon = -$:

$$R\colon L' \otimes_{\mathbb{Z}} Q \to L'_\circ \otimes_{\mathbb{Z}} Q, \qquad (a_1, a_2) \mapsto (a_1, 0, a_2).$$

Here the subtuple a_1 corresponds to the vertices of Γ consisting of the vertex labeled by m_1 and all vertices on its left. The subtuple a_2 corresponds to the vertices of Γ consisting of the vertex labeled by m_2 and all vertices on its right. The 0 entry corresponds to the added vertex in $\Gamma_\circ$. (See Remark 1.3 about the determination of left and right parts of the trees.) For $a \in \delta + 2L' \otimes_{\mathbb{Z}} Q$, define $a_\circ := R(a) \in \delta_\circ + 2L'_\circ \otimes_{\mathbb{Z}} Q$.

Since $\Gamma_\circ$ does not have a new vertex of degree ≥ 3, nor has it a new forcing bridge, the top and bottom plumbing trees have isomorphic sets of Weyl assignments.

The added vertex in $\Gamma_\circ$ has degree 2. From (4.1), we deduce that for $\ell_\circ \in a_\circ + 2B_\circ L_\circ \otimes_{\mathbb{Z}} Q$, one has $c_{\Gamma_\circ}(\ell_\circ) = 0$ when the component of $\ell_\circ$ corresponding to the added vertex is non-zero. Hence, we can restrict the sum in the series for $\Gamma_\circ$ over only those $\ell_\circ$ which are of type $\ell_\circ = R(\ell)$ for some $\ell \in a + 2BL \otimes_{\mathbb{Z}} Q$. As R is injective, it will be enough to verify that the contribution of $\ell \in a + 2BL \otimes_{\mathbb{Z}} Q$ in the series for Γ equals the contribution of $R(\ell)$ in the series for $\Gamma_\circ$.

From (4.1), one has $c_{\Gamma_\circ}(R(\ell)) = c_\Gamma(\ell)$. Moreover, a direct computation shows that

$$B^{-1}\ell = (h_1, h_2) \quad \Rightarrow \quad B_\circ^{-1} R(\ell) = (h_1, h_0, h_2)$$

for some h_0. (Specifically, h_0 is the sum of the entry of h_1 and the entry of h_2 corresponding to the two vertices adjacent to the added vertex in $\Gamma_\circ$. However, the explicit expression of h_0 will not be needed below.) This implies that writing $\ell = (\ell_1, \ell_2)$, one has

$$\langle R(\ell), R(\ell)\rangle = (\ell_1, 0, \ell_2)^t (h_1, h_0, h_2) = \langle \ell, \ell\rangle. \tag{5.2}$$

We conclude that

$$c_\Gamma(\ell)\, q^{-\frac{1}{8}\langle \ell,\ell\rangle} = c_{\Gamma_\circ}(R(\ell))\, q^{-\frac{1}{8}\langle R(\ell), R(\ell)\rangle}. \tag{5.3}$$

Hence the contribution of ℓ in the series for Γ equals the contribution of $R(\ell)$ in the series for $\Gamma_\circ$. This implies the statement for this move.

Step (A+): The Neumann move (A+) from Figure 2. In this case, one has

$$\sigma_\circ = \sigma + 1, \qquad \pi_\circ = 1 + \pi, \qquad 3\sigma_\circ - \operatorname{tr} B_\circ = 3\sigma - \operatorname{tr} B.$$

We conclude that the factor (5.1) in front of the sum in the series for $\Gamma_\circ$ has an extra factor $(-1)^{|\Delta^+|}$.

Next, we use the function R from (1.6) this time with $\epsilon = +$:

$$R \colon L' \otimes_{\mathbb{Z}} Q \to L'_\circ \otimes_{\mathbb{Z}} Q, \qquad (a_1, a_2) \mapsto (a_1, 0, -a_2).$$

For $a \in \delta + 2L' \otimes_{\mathbb{Z}} Q$, define $a_\circ := R(a) \in \delta_\circ + 2L'_\circ \otimes_{\mathbb{Z}} Q$.

As for the previous move, the sets of Weyl assignments for the two plumbing trees are isomorphic. The natural isomorphism is defined as follows. For a Weyl assignment ξ for the bottom plumbing tree, define a Weyl assignment $\xi_\circ$ for the top plumbing tree such that for a vertex v with $\deg v \geq 3$, one has

$$\xi_\circ \colon v \mapsto \begin{cases} \xi_v & \text{if } v \text{ is on the left of the added vertex,} \\ \iota\xi_v & \text{if } v \text{ is on the right of the added vertex.} \end{cases} \tag{5.4}$$

Here, $\iota \in W$ is as in (2.7). Since the added vertex has degree 2, the value of $\xi_\circ$ at the added vertex is 1_W, as determined by (3.2).

Note that when the added vertex is on a forcing bridge $\Gamma_{v,w}$, the definition of $\xi_\circ$ via (5.4) is compatible with the condition (3.3), since

$$\Delta\pi_\circ(v, w) = \Delta\pi(v, w) + 1$$

where $\Delta\pi(v, w)$ and $\Delta\pi_\circ(v, w)$ are the differences in numbers of positive eigenvalues obtained from the contraction of the bridge $\Gamma_{v,w}$ in Γ and $\Gamma_\circ$, respectively, as in (3.4).

As for the previous move, we can restrict the sum in the series for $\Gamma_\circ$ over only those $\ell_\circ$ which are of type $\ell_\circ = R(\ell)$ for some $\ell \in a + 2BL \otimes_{\mathbb{Z}} Q$. In this case, we have

$$c_\Gamma(\ell) = (-1)^{|\Delta^+|} c_{\Gamma_\circ}(R(\ell)). \tag{5.5}$$

This follows from Lemma 4.3, the definition of $\xi_\circ$, and the fact that

$$\prod_{v \in V_2(\Gamma_\circ)} (-1)^{\deg v} = -1 \tag{5.6}$$

where $V_2(\Gamma_\circ)$ is the set of all vertices of $\Gamma_\circ$ on the right of the added vertex. Indeed, one has that $\sum_{v \in V_2(\Gamma_\circ)} \deg v$ is odd, since every edge on the right of the added vertex in $\Gamma_\circ$ is incident to two vertices in $V_2(\Gamma_\circ)$ with the exception of the edge incident to the added vertex, which is incident to only one vertex in $V_2(\Gamma_\circ)$. The factor $(-1)^{|\Delta^+|}$ in (5.5) matches the extra contribution to the factor (5.1) in front of the sum in the series for $\Gamma_\circ$. That is, we have

$$(-1)^{|\Delta^+|\pi} c_\Gamma(\ell) = (-1)^{|\Delta^+|\pi_\circ} c_{\Gamma_\circ}(R(\ell)).$$

A direct computation shows that

$$B^{-1}\ell = (h_1, h_2) \quad \Rightarrow \quad B_\circ^{-1} R(\ell) = (h_1, h_0, -h_2)$$

for some h_0. (Specifically, h_0 is minus the sum of the entry of h_1 and the entry of $-h_2$ corresponding to the two vertices adjacent to the added vertex in $\Gamma_\circ$; however, the formula for h_0 will not be needed below.) This implies that

$$\langle R(\ell), R(\ell) \rangle = (\ell_1, 0, -\ell_2)^t (h_1, h_0, -h_2) = \langle \ell, \ell \rangle. \tag{5.7}$$

We conclude that

$$(-1)^{|\Delta^+|\pi} c_\Gamma(\ell)\, q^{-\frac{1}{8}\langle \ell, \ell \rangle} = (-1)^{|\Delta^+|\pi_\circ} c_{\Gamma_\circ}(R(\ell))\, q^{-\frac{1}{8}\langle R(\ell), R(\ell) \rangle}. \tag{5.8}$$

Hence the contribution of ℓ in the series for Γ equals the contribution of $R(\ell)$ in the series for $\Gamma_\circ$. Since R is injective, the statement for this move follows.

Step (B−): The Neumann move (B−) from Figure 2. In this case, one has

$$\sigma_\circ = \sigma - 1, \qquad \pi_\circ = \pi, \qquad 3\sigma_\circ - \operatorname{tr} B_\circ = -1 + 3\sigma - \operatorname{tr} B.$$

We conclude that the factor (5.1) in front of the sum in the series for $\Gamma_\circ$ has an extra factor $q^{-\frac{1}{2}\langle \rho, \rho \rangle}$.

For a choice of $w \in W$, consider the function:

$$R_w \colon L' \otimes_{\mathbb{Z}} Q \to L'_\circ \otimes_{\mathbb{Z}} Q, \qquad (a_\sharp, a_1) \mapsto (a_\sharp, a_1 + 2w(\rho), -2w(\rho)) \tag{5.9}$$

with entry a_1 corresponding to the vertex of Γ labeled by m_1, subtuple $a_\sharp$ corresponding to all other vertices of Γ, and entry $-2w(\rho)$ on the right-hand side corresponding to the added vertex in $\Gamma_\circ$. Note that the function R from (1.7) with $\epsilon = -$ is $R = R_w$ with $w = 1_W$. For $a \in \delta + 2L' \otimes_{\mathbb{Z}} Q$, define $a_\circ := R(a) \in \delta_\circ + 2L'_\circ \otimes_{\mathbb{Z}} Q$.

The added vertex in $\Gamma_\circ$ has degree 1. From (4.2), we deduce that for $\ell_\circ \in a_\circ + 2B_\circ L_\circ \otimes_{\mathbb{Z}} Q$, one has $c_{\Gamma_\circ}(l_\circ) = 0$ when the component of $\ell_\circ$ corresponding to the added vertex is not in the orbit $-2W(\rho)$. Hence, we can restrict the sum in the series for $\Gamma_\circ$ over only those $\ell_\circ$ which are of type $\ell_\circ = R_w(\ell)$ for some $\ell \in a + 2BL \otimes_{\mathbb{Z}} Q$ and some $w \in W$. Note that for $\ell \in a + 2BL \otimes_{\mathbb{Z}} Q$ and $w \in W$, one indeed has

$$R_w(\ell) \in a_\circ + 2B_\circ L_\circ \otimes_{\mathbb{Z}} Q.$$

Let n be the degree of the vertex in $\Gamma_\circ$ adjacent to the added vertex. Then the degree of the corresponding vertex in Γ is $n-1$. The assumption that this Neumann move does not create a reducible vertex implies $n \neq 3$. It follows that $\Gamma_\circ$ does not have a new vertex of degree at least 3, nor has it a new forcing bridge, hence the top and bottom plumbing trees have isomorphic sets of Weyl assignments; denote these as Ξ.

For $\ell \in a + 2BL \otimes_{\mathbb{Z}} Q$, write $\ell = (\ell_\sharp, \ell_1)$. Select $\xi \in \Xi$, and let $x \in W$ be the value of ξ at the vertex in $\Gamma_\circ$ adjacent to the added vertex. From Lemma 4.2, one has

$$\begin{aligned} c_{x,n-1}(\ell_1) &= (-1)^{|\Delta^+|} \sum_{w\in W} (-1)^{\ell(w)}\, c_{x,n}\left(\ell_1 + 2w(\rho)\right) \\ &= \sum_{w\in W} c_{1_W,1}\left(-2w(\rho)\right) c_{x,n}\left(\ell_1 + 2w(\rho)\right). \end{aligned} \tag{5.10}$$

The second identity follows from equation (4.2) and Lemma 2.4, which together imply

$$c_{1_W,1}\left(-2w(\rho)\right) = (-1)^{\ell(\iota w)} = (-1)^{|\Delta^+|}(-1)^{\ell(w)} \qquad \text{for } w \in W.$$

Multiplying both sides of (5.10) by the contributions corresponding to the remaining vertices of Γ, one has

$$\prod_{v\in V(\Gamma)} \left[P^{\xi_v}_{\deg v}(z_v)\right]_{\ell_v} = \sum_{w\in W} \prod_{v\in V(\Gamma_\circ)} \left[P^{\xi_v}_{\deg v}(z_v)\right]_{R_w(\ell)_v}. \tag{5.11}$$

Averaging over all $\xi \in \Xi$, one has

$$c_\Gamma(\ell) = \sum_{w\in W} c_{\Gamma_\circ}(R_w(\ell)). \tag{5.12}$$

Next, we consider the powers of q. For $w \in W$, a direct computation shows that

$$B^{-1}\ell = (h_\sharp, h_1) \quad \Rightarrow \quad B_\circ^{-1} R_w(\ell) = \left(h_\sharp, h_1, h_1 + 2w(\rho)\right).$$

This implies that

$$\begin{aligned} \langle R_w(\ell), R_w(\ell)\rangle &= \left(\ell_\sharp, \ell_1 + 2w(\rho), -2w(\rho)\right)^t \left(h_\sharp, h_1, h_1 + 2w(\rho)\right) \\ &= \langle \ell, \ell\rangle - 4\langle \rho, \rho\rangle. \end{aligned}$$

Thus from (5.12), we have

$$c_\Gamma(\ell)\, q^{-\frac{1}{8}\langle \ell,\ell\rangle} = q^{-\frac{1}{2}\langle \rho,\rho\rangle} \sum_{w\in W} c_{\Gamma_\circ}(R_w(\ell))\, q^{-\frac{1}{8}\langle R_w(\ell),R_w(\ell)\rangle}. \tag{5.13}$$

The factor $q^{-\frac{1}{2}\langle \rho,\rho\rangle}$ on the right-hand side matches the extra contribution to the factor (5.1) in front of the sum in the series for $\Gamma_\circ$. We conclude that the contribution of ℓ in the series for Γ equals the sum over $w \in W$ of the contributions of $R_w(\ell)$ in the series for $\Gamma_\circ$. Since the maps R_w for $w \in W$ are injective, the statement for this move follows.

Step (B+): The Neumann move (B+) from Figure 2. In this case, one has

$$\sigma_\circ = 1+\sigma, \qquad \pi_\circ = 1+\pi, \qquad 3\sigma_\circ - \operatorname{tr} B_\circ = 1 + 3\sigma - \operatorname{tr} B.$$

We conclude that the factor (5.1) in front of the sum in the series for $\Gamma_\circ$ has an extra factor $(-1)^{|\Delta^+|} q^{\frac{1}{2}\langle \rho,\rho\rangle}$.

For a choice of $w \in W$, consider the function

$$R_w \colon L' \otimes_{\mathbb{Z}} Q \to L'_\circ \otimes_{\mathbb{Z}} Q, \qquad (a_\sharp, a_1) \mapsto (a_\sharp, a_1 + 2w(\rho), 2w(\rho)). \tag{5.14}$$

For $a \in \delta + 2L' \otimes_{\mathbb{Z}} Q$, define $a_\circ := R(a) \in \delta_\circ + 2L'_\circ \otimes_{\mathbb{Z}} Q$ where $R = R_w$ with $w = 1_W$.

As with the previous move, we can restrict the sum in the series for $\Gamma_\circ$ over only those $\ell_\circ$ which are of type $\ell_\circ = R_w(\ell)$ for some $\ell \in a + 2BL \otimes_{\mathbb{Z}} Q$ and some $w \in W$. Also, let n be the degree of the vertex in $\Gamma_\circ$ adjacent to the added vertex. As with the previous move, the assumption that this Neumann move does not create a reducible vertex implies $n \neq 3$, and thus there are no new forcing bridges. Hence the sets of Weyl assignments for the two plumbing trees are isomorphic.

For $\ell \in a + 2BL \otimes_{\mathbb{Z}} Q$, write $\ell = (\ell_\sharp, \ell_1)$. Select $\xi \in \Xi$, and let $x \in W$ be the value of ξ at the vertex in $\Gamma_\circ$ adjacent to the added vertex. From Lemma 4.2, one has

$$\begin{aligned} c_{x,n-1}(\ell_1) &= (-1)^{|\Delta^+|} \sum_{w\in W} (-1)^{\ell(w)}\, c_{x,n}\left(\ell_1 + 2w(\rho)\right) \\ &= (-1)^{|\Delta^+|} \sum_{w\in W} c_{1_W,1}\left(2w(\rho)\right) c_{x,n}\left(\ell_1 + 2w(\rho)\right). \end{aligned} \tag{5.15}$$

The second identity follows from equation (4.2). Multiplying both sides of (5.15) by the contributions corresponding to the remaining vertices of Γ, one has

$$\prod_{v\in V(\Gamma)} \left[P^{\xi_v}_{\deg v}(z_v)\right]_{\ell_v} = (-1)^{|\Delta^+|} \sum_{w\in W} \prod_{v\in V(\Gamma_\circ)} \left[P^{\xi_v}_{\deg v}(z_v)\right]_{R_w(\ell)_v}. \tag{5.16}$$

Averaging over all $\xi \in \Xi$, one has

$$c_\Gamma(\ell) = (-1)^{|\Delta^+|} \sum_{w\in W} c_{\Gamma_\circ}(R_w(\ell)). \tag{5.17}$$

For $w \in W$, a direct computation shows that

$$B^{-1}\ell = (h_\sharp, h_1) \quad \Rightarrow \quad B_\circ^{-1} R_w(\ell) = (h_\sharp, h_1, 2w(\rho) - h_1)\,.$$

This implies that

$$\begin{aligned}\langle R_w(\ell), R_w(\ell)\rangle &= (\ell_\sharp, \ell_1 + 2w(\rho), 2w(\rho))^t\,(h_\sharp, h_1, 2w(\rho) - h_1)\\ &= \langle \ell, \ell\rangle + 4\langle \rho, \rho\rangle.\end{aligned}$$

Thus from (5.17), we have

(5.18)
$$(-1)^{|\Delta^+|\,\pi} c_\Gamma(\ell)\, q^{-\frac{1}{8}\langle \ell,\ell\rangle} = (-1)^{|\Delta^+|\,\pi_\circ} q^{\frac{1}{2}\langle\rho,\rho\rangle} \sum_{w\in W} c_{\Gamma_\circ}(R_w(\ell))\, q^{-\frac{1}{8}\langle R_w(\ell), R_w(\ell)\rangle}.$$

The factor $q^{\frac{1}{2}\langle\rho,\rho\rangle}$ on the right-hand side matches the extra contribution to q in the factor (5.1) in front of the sum in the series for $\Gamma_\circ$. We conclude that the contribution of ℓ in the series for Γ equals the sum over $w \in W$ of the contributions of $R_w(\ell)$ in the series for $\Gamma_\circ$. Since the maps R_w for $w \in W$ are injective, the statement for this move follows.

Step (C): The Neumann move (C) from Figure 2. In this case, one has

$$\sigma_\circ = \sigma, \qquad \pi_\circ = 1 + \pi, \qquad 3\sigma_\circ - \operatorname{tr} B_\circ = 3\sigma - \operatorname{tr} B.$$

We conclude that the factor (5.1) in front of the sum in the series for $\Gamma_\circ$ has an extra factor $(-1)^{|\Delta^+|}$.

Recall the function R_β with $\beta \in Q$ from (1.8):

$$R_\beta\colon L' \otimes_{\mathbb{Z}} Q \to L'_\circ \otimes_{\mathbb{Z}} Q, \qquad (a_\sharp, a_0, a_\flat) \mapsto (a_\sharp, a_0 + \beta, 0, \beta, -a_\flat)$$

where the entry a_0 corresponds to the vertex in Γ labelled by $m_1 + m_2$, the entries $a_0 + \beta$, 0, and β correspond to the vertices in $\Gamma_\circ$ labelled by m_1, 0, and m_2, respectively, and the subtuples $a_\sharp$ and $a_\flat$ correspond to all the vertices in $\Gamma_\circ$ on their left and right, respectively.

For $a \in \delta + 2L' \otimes_{\mathbb{Z}} Q$, define $a_\circ \in \delta_\circ + 2L'_\circ \otimes_{\mathbb{Z}} Q$ as

$$a_\circ := \begin{cases} R_{2\rho}(a) & \text{if } \deg(v_1) \equiv \deg(v_2) \bmod 2,\\ R_0(a) & \text{otherwise.}\end{cases}$$

This is as in (1.9).

As the vertex labelled by 0 in $\Gamma_\circ$ has degree 2, from (4.1) we deduce that for $\ell_\circ \in a_\circ + 2B_\circ L_\circ \otimes_{\mathbb{Z}} Q$, one has $c_{\Gamma_\circ}(\ell_\circ) = 0$ when $\ell_\circ$ has a non-zero component corresponding to the vertex of $\Gamma_\circ$ labelled by 0. Hence, we can restrict the sum in the series for $\Gamma_\circ$ over only those $\ell_\circ$ which are of type $\ell_\circ = R_\beta(\ell)$ for some $\ell \in a + 2BL \otimes_{\mathbb{Z}} Q$ and some $\beta \in Q$. Note that for $\ell \in a + 2BL \otimes_{\mathbb{Z}} Q$ and $\beta \in Q$, one has

$$R_\beta(\ell) \in a_\circ + 2B_\circ L_\circ \otimes_{\mathbb{Z}} Q$$

if and only if $\beta \in \beta_0 + 2Q$ with β_0 defined as in (1.9).

Select a Weyl assignment ξ for Γ, and define a Weyl assignment $\xi_\circ$ for $\Gamma_\circ$ such that, for a vertex v with $\deg v \geq 3$, one has

$$\xi_\circ \colon v \mapsto \begin{cases} \xi_v & \text{if } v \text{ is on the left of } v_1, \\ \iota\xi_v & \text{if } v \text{ is on the right of } v_2. \end{cases}$$

Here ι is as in (2.7), and v_1 and v_2 are the vertices labelled by m_1 and m_2 in $\Gamma_\circ$. Moreover, define

$$\xi_\circ(v_1) := \xi(v_0) \qquad \text{and} \qquad \xi_\circ(v_2) := \iota\xi(v_0)$$

where v_0 is the vertex labelled by $m_1 + m_2$ in Γ. The value of $\xi_\circ$ at the vertex labelled by 0 in $\Gamma_\circ$ is 1_W, as determined by (3.2). The map $\xi \mapsto \xi_\circ$ is the natural isomorphism of the sets of Weyl assignments for Γ and $\Gamma_\circ$.

Let p and q be the degrees of the vertices v_1 and v_2 in $\Gamma_\circ$, respectively. Then the degree of the vertex v_0 in Γ is $p+q-2$. Recall that we are assuming that the Neumann move does not create a new reducible vertex. Moreover, when $p, q \geq 3$, the tree $\Gamma_\circ$ has one more forcing bridge with respect to Γ, and the definition of $\xi_\circ$ is compatible with the condition (3.3).

For $\ell \in a + 2BL \otimes_{\mathbb{Z}} Q$, write $\ell = (\ell_\sharp, \ell_0, \ell_\flat)$. Let $x := \xi(v_0) \in W$. From Lemma 4.5, one has

$$c_{x,p+q-2}(\ell_0) = \sum_{\beta \in \beta_0 + 2Q} c_{x,p}\left(\ell_0 + \beta\right) c_{x,q}\left(-\beta\right).$$

Applying Lemma 4.3, one has

$$c_{x,p+q-2}(\ell_0) = (-1)^{|\Delta^+|\, q} \sum_{\beta \in \beta_0 + 2Q} c_{x,p}\left(\ell_0 + \beta\right) c_{\iota x,q}\left(\beta\right). \tag{5.19}$$

Let $V_\flat(\Gamma_\circ)$ be the set of all vertices of $\Gamma_\circ$ on the right of the vertex labelled by m_2. Applying Lemma 4.3 to all contributions corresponding to vertices in $V_\flat(\Gamma_\circ)$ and using that $q + \sum_{v \in V_\flat(\Gamma_\circ)} \deg v$ is odd (this is as in (5.6)), one has

$$(-1)^{|\Delta^+|\, q} c_{\iota x,q}\left(\beta\right) \prod_{v \in \mathrm{V}_\flat(\Gamma_\circ)} c_{\xi_v, \deg v}(\ell_v) = (-1)^{|\Delta^+|} c_{\iota x,q}\left(\beta\right) \prod_{v \in \mathrm{V}_\flat(\Gamma_\circ)} c_{\iota\xi_v, \deg v}(-\ell_v).$$

Multiplying both sides of (5.19) by the contributions corresponding to the remaining vertices, using (4.1) and the last identity, one has

$$\prod_{v \in V(\Gamma)} \left[P^{\xi_v}_{\deg v}(z_v)\right]_{\ell_v} = (-1)^{|\Delta^+|} \sum_{\beta \in \beta_0 + 2Q} \prod_{v \in V(\Gamma_\circ)} \left[P^{\xi_\circ(v)}_{\deg v}(z_v)\right]_{R_\beta(\ell)_v}. \tag{5.20}$$

Averaging over all $\xi \in \Xi$, one has

$$c_\Gamma(\ell) = (-1)^{|\Delta^+|} \sum_{\beta \in \beta_0 + 2Q} c_{\Gamma_\circ}(R_\beta(\ell)). \tag{5.21}$$

The factor $(-1)^{|\Delta^+|}$ on the right-hand side matches the extra contribution to the factor (5.1) in front of the sum in the series for $\Gamma_\circ$.

For $\beta \in Q$, a direct computation shows that

$$B^{-1}\ell = (h_\sharp, h_0, h_\flat) \quad \Rightarrow \quad B_\circ^{-1} R_\beta(\ell) = \big(h_\sharp, h_0, h_0', -h_0, -h_\flat\big)$$

for some $h_0' \in Q$. This implies that

$$\langle R_\beta(\ell), R_\beta(\ell)\rangle = (\ell_\sharp, \ell_0 + \beta, 0, \beta, -\ell_\flat)^t \big(h_\sharp, h_0, h_0', -h_0, -h_\flat\big) = \langle \ell, \ell\rangle. \tag{5.22}$$

We conclude that

$$(-1)^{|\Delta^+|\,\pi} c_\Gamma(\ell) q^{-\frac{1}{8}\langle \ell,\ell\rangle} = (-1)^{|\Delta^+|\,\pi_\circ} \sum_{\beta \in \beta_0 + 2Q} c_{\Gamma_\circ}(R_\beta(\ell)) q^{-\frac{1}{8}\langle R_w(\ell), R_w(\ell)\rangle}. \tag{5.23}$$

Hence the contribution of ℓ in the series for Γ equals the sum over $\beta \in \beta_0+2Q$ of the contributions of $R_\beta(\ell)$ in the series for $\Gamma_\circ$. Since the maps R_β are injective for all β, the statement for this move follows.

This concludes the proof. □

Finally, we prove:

Proof of Theorem 1. Part (i) follows from Theorem 3.4 and [Neu] (see §1.5); part (ii) follows from Theorem 5.1. □

6. Brieskorn spheres

Here we prove Corollary 2 regarding the computation of the invariant series for a Brieskorn homology sphere $\Sigma(b_1, b_2, b_3)$. For this 3-manifold, the series $\widehat{Z}(q)$ was first computed for $Q = A_1$ in [GM] and then for arbitrary Q in [Par]. Here we show how these computations can be extended more generally to the case of the series $\mathsf{Y}_{P,a}(q)$.

For integers b_1, b_2, b_3, the Brieskorn sphere $M = \Sigma(b_1, b_2, b_3)$ is the link of the singularity $z_1^{b_1} + z_2^{b_2} + z_3^{b_3} = 0$, and it is a homology sphere if and only if b_1, b_2, b_3 are pairwise coprime. If any of the b_1, b_2, b_3 is equal to 1, then $\Sigma(b_1, b_2, b_3)$ is homeomorphic to the 3-sphere. Hence, we will assume that b_1, b_2, b_3 are pairwise coprime and $2 \le b_1 < b_2 < b_3$.

The Brieskorn homology sphere $M = \Sigma(b_1, b_2, b_3)$ is realized as a negative definite plumbed manifold with plumbing tree Γ defined as follows. Select integers $b < 0$ and $a_1, a_2, a_3 > 0$ such that

$$b + \sum_{i=1}^{3} \frac{a_i}{b_i} = -\frac{1}{b_1 b_2 b_3}.$$

Then Γ is a star-shaped tree consisting of a central vertex of degree 3 labelled by b and three legs consisting of vertices labelled by $-k_1^i, \ldots, -k_{s_i}^i$, for $i = 1, 2, 3$, such that for each leg, the labels are ordered starting from the central vertex and

$$\frac{b_i}{a_i} = k_1^i - \cfrac{1}{k_2^i - \cfrac{1}{\ddots - \cfrac{1}{k_{s_i}^i}}}.$$

The total number of vertices is $s = 1 + s_1 + s_2 + s_3$. Since Γ is negative definite, one has $\pi = 0$ and $\sigma = -s$. As Γ has only one vertex of degree ≥ 3, the set of Weyl assignments from §3.2 is $\Xi \cong W$.

Select a root lattice Q, and choose an order of the basis of $L' \otimes_{\mathbb{Z}} Q \cong Q^s$ so that for $f \in Q^s$ one writes

$$f = (f_0, f_1, f_2, f_3, \dots) \tag{6.1}$$

with f_0 corresponding to the vertex of Γ of degree 3 and f_1, f_2, f_3 to the three vertices of degree 1.

Since $H_1(M; Q) = 0$, one has that $a = \delta$ from (1.2) is the unique generalized Spin^c-structure. This is $\delta = (-1, 1, 1, 1, 0 \dots, 0) \otimes 2\rho$.

Proof of Corollary 2. For an admissible series $P(z) = \sum_{\alpha \in Q} c(\alpha) z^\alpha$, the series from (3.6) is

$$\mathsf{Y}_P(q) = q^{-\frac{1}{2}(3s + \operatorname{tr} B)\langle \rho, \rho \rangle} \sum_f c_\Gamma(f)\, q^{-\frac{1}{8}\langle f, f \rangle} \tag{6.2}$$

where the sum is over $f \in \delta + 2BL \otimes Q \subset Q^s$. Recall the definition of the coefficients $c_\Gamma(f)$ in (3.7) in terms of the graded Weyl twists from §3.1. Since Γ is a star-shaped tree with three legs, the sum over f in (6.2) can be restricted to those f which are of the form

$$f = (\alpha, 2w_1(\rho), 2w_2(\rho), 2w_3(\rho), 0, \dots, 0) \in Q^s$$
$$\text{with } \alpha \in -2\rho + 2Q \text{ and } w_1, w_2, w_3 \in W$$

as $c_\Gamma(f)$ vanishes otherwise. For a fixed $x \in W \cong \Xi$, applying (3.1) with $n = 3$, one has that the contribution of $f_0 = \alpha$ is $(-1)^{\ell(x)} c(x^{-1}(\alpha))$, and applying (4.2), the contribution of $f_i = 2w_i(\rho)$ for $i = 1, 2, 3$ is $(-1)^{\ell(w_i)}$. Hence one has

$$c_\Gamma(f) = \frac{1}{|W|} \sum_{x \in W} (-1)^{\ell(x w_1 w_2 w_3)} c\left(x^{-1}\alpha\right).$$

Thus (6.2) can be rewritten as

$$\mathsf{Y}_P(q) = q^{-\frac{1}{2}(3s + \operatorname{tr} B)\langle \rho, \rho \rangle} \sum_f \frac{1}{|W|} \sum_{x \in W} (-1)^{\ell(x w_1 w_2 w_3)} c\left(x^{-1}\alpha\right) q^{-\frac{1}{8}\langle f, f \rangle}$$

Since $\ell(x) = \ell(x^{-1})$, one has $\ell(x w_1 w_2 w_3) \equiv \ell(x^{-1} w_1 x^{-1} w_2 x^{-1} w_3) \bmod 2$. Thus by making use of the symmetry $f \mapsto x^{-1} f$ and the fact that $\langle f, f \rangle = \langle x^{-1} f, x^{-1} f \rangle$, the series becomes

$$\mathsf{Y}_P(q) = q^{-\frac{1}{2}(3s + \operatorname{tr} B)\langle \rho, \rho \rangle} \sum_f (-1)^{\ell(w_1 w_2 w_3)} c\left(\alpha\right) q^{-\frac{1}{8}\langle f, f \rangle}.$$

To further simplify it, let us analyze the exponents of q. From the definition of the pairing of $L' \otimes_{\mathbb{Z}} Q \cong Q^s$ in (1.1), one has

$$\langle f, f\rangle = B^{-1}_{00}\langle\alpha,\alpha\rangle + 2\sum_{i=1}^{3} B^{-1}_{0i}\langle\alpha, 2w_i(\rho)\rangle + \sum_{i,j=1}^{3} B^{-1}_{ij}\langle 2w_i(\rho), 2w_j(\rho)\rangle \tag{6.3}$$

where the entries of B^{-1} are indexed by $i, j = 0, \ldots, s-1$, compatibly with the indices of f in (6.1). In particular, only the top 4×4 diagonal block of B^{-1} is required here. A direct computation in [GM] shows that the related entries of B^{-1} are as follows

$$B^{-1}_{00} = -b_1b_2b_3, \qquad B^{-1}_{ij} = -\frac{b_1b_2b_3}{b_ib_j} \quad \text{for } i,j = 1,2,3 \text{ with } i\neq j,$$
$$B^{-1}_{0i} = -\frac{b_1b_2b_3}{b_i} \quad \text{and} \quad B^{-1}_{ii} = -h_i \quad \text{for } i = 1,2,3,$$

where $h_i > 0$ is (up to a sign) the determinant of the framing matrix for the plumbing graph obtained from Γ by deleting the terminal vertex on the ith leg (the exact value of h_i will not be needed in the following). Substituting in (6.3) yields

$$\langle f, f\rangle = -b_1b_2b_3\left|\left|\alpha + \sum_{i=1}^{3}\frac{2w_i(\rho)}{b_i}\right|\right|^2 + 4\langle\rho,\rho\rangle\left(b_1b_2b_3\sum_{i=1}^{3}\frac{1}{b_i^2} - \sum_{i=1}^{3} h_i\right).$$

Hence, we can rewrite the series as

$$\mathsf{Y}_P(q) = q^C \sum_f (-1)^{\ell(w_1w_2w_3)} c\,(\alpha)\, q^{e(f)} \tag{6.4}$$

where

$$C := -\frac{1}{2}\left(3s + \operatorname{tr} B + b_1b_2b_3\sum_{i=1}^{3}\frac{1}{b_i^2} - \sum_{i=1}^{3} h_i\right)\langle\rho,\rho\rangle \in \mathbb{Q} \tag{6.5}$$

and

$$e(f) := \frac{b_1b_2b_3}{8}\left|\left|\alpha + \sum_{i=1}^{3}\frac{2w_i(\rho)}{b_i}\right|\right|^2 \in \mathbb{Q}.$$

For $d := b_1b_2b_3$ and $w := w_3$, one has

$$e(f) = \frac{1}{8d}\left|\left|d\alpha + w(\eta_{x,y})\right|\right|^2$$

where $x := w^{-1}w_1$, $y := w^{-1}w_2$, and $\eta_{x,y}$ is as in (0.1). Decomposing the sum over f as a sum over α and w_i, for $i = 1,2,3$, the series (6.4) is rearranged as in the statement. $\square$

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References

[AJK] R. Akhmechet, P. K. Johnson, and V. Krushkal. Lattice cohomology and q-series invariants of 3-manifolds. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2023(796):269–299, 2023. ↑1, ↑2, ↑5, ↑37

[BMM1] K. Bringmann, K. Mahlburg, and A. Milas. Higher depth quantum modular forms and plumbed 3-manifolds. *Letters in Mathematical Physics*, 110(10):2675–2702, 2020. ↑5

[BMM2] K. Bringmann, K. Mahlburg, and A. Milas. Quantum modular forms and plumbing graphs of 3-manifolds. *Journal of Combinatorial Theory, Series A*, 170:105145, 2020. ↑5

[Bou] N. Bourbaki. *Lie groups and Lie algebras. Chapters 4–6*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2002. Translated from the 1968 French original by Andrew Pressley. ↑5

[Chu] H.-J. Chung. BPS invariants for Seifert manifolds. *Journal of High Energy Physics*, 2020(3):1–67, 2020. ↑3

[GM] S. Gukov and C. Manolescu. A two-variable series for knot complements. *Quantum Topology*, 12(1), 2021. ↑1, ↑2, ↑3, ↑4, ↑20, ↑21, ↑22, ↑23, ↑34, ↑36, ↑37

[GPPV] S. Gukov, D. Pei, P. Putrov, and C. Vafa. BPS spectra and 3-manifold invariants. *Journal of Knot Theory and Its Ramifications*, 29(02):2040003, 2020. ↑1, ↑2, ↑3, ↑23

[GPV] S. Gukov, P. Putrov, and C. Vafa. Fivebranes and 3-manifold homology. *Journal of High Energy Physics*, 2017(7):1–82, 2017. ↑1

[Hum] J. E. Humphreys. *Introduction to Lie algebras and representation theory*, volume Vol. 9 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1972. ↑5, ↑19

[LM] L. Liles and E. McSpirit. Infinite families of quantum modular 3-manifold invariants. *Communications in Number Theory and Physics, to appear, arXiv:2306.14765*, 2023. ↑2

[LZ] R. Lawrence and D. Zagier. Modular forms and quantum invariants of 3-manifolds. *Asian Journal of Mathematics*, 3(1):93–108, 1999. ↑3, ↑4

[Ném] A. Némethi. *Normal surface singularities*, volume 74 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer, 2022. ↑6, ↑10

[Neu] W. D. Neumann. A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves. *Transactions of the American Mathematical Society*, 268(2):299–344, 1981. ↑6, ↑9, ↑11, ↑34

[Par] S. Park. Higher rank $\widehat{Z}$ and F_K. *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications*, 16:044, 2020. ↑2, ↑3, ↑4, ↑5, ↑9, ↑20, ↑21, ↑23, ↑34, ↑37

[Ri] S. J. Ri. Refined and generalized $\hat{Z}$ invariants for plumbed 3-manifolds. *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications*, 19:011, 2023. ↑2, ↑9, ↑12, ↑13, ↑20, ↑22, ↑26, ↑37
[Zag] D. Zagier. Quantum modular forms. *Quanta of maths*, 11:659–675, 2010. ↑5

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