SIGNATURE, SLICING FOAMS, AND CROSSING CHANGES OF KLEIN GRAPHS

KENNETH L. BAKER, ALLISON H. MOORE, DANIELLE O'DONNOL, AND SCOTT TAYLOR

ABSTRACT. A totally oriented Klein graph is a trivalent spatial graph in the 3-sphere with a 3-coloring of its edges and an orientation on each bicolored link. A totally oriented Klein foam is a 3-colored 2-complex in the 4-ball whose boundary is a Klein foam and whose bicolored surfaces are oriented. We extend Gille-Robert's signature for 3-Hamiltonian Klein graphs to all totally oriented Klein graphs and develop an analogy of Murasugi's bounds relating the signature, slice genus and unknotting number of knots. In particular, we show that the signature of a totally oriented Klein graph produces a lower bound on the negative orbifold Euler characteristic of certain totally oriented Klein foams bounded by Γ. When Γ is abstractly planar, these negative Euler characteristics, in turn, produce a lower bound on a certain natural unknotting number for Γ. *Mutatis mutandi*, we produce lower bounds on the corresponding Gordian distance between two totally oriented Klein graphs that can be related by a sequence of crossing changes. We also give examples of θ-curves for which our lower bounds on unknotting number improve on previously known bounds.

1. INTRODUCTION

For a knot $K \subset S^3$, the familiar chain of inequalities

$$\frac{1}{2}|\sigma(K)| \leq g_4(K) \leq u(K)$$

between the signature $\sigma(K)$, slice genus $g_4(K)$, and unknotting number $u(K)$ [Mur65, Theorem 9.1 and Theorem 10.1] can be rewritten as

$$|\sigma(K)| \leq 1 - \chi_4(K) \leq 2u(K).$$

Here $\chi_4(K)$ is the *slice Euler characteristic*, the maximal Euler characteristic of a properly embedded, orientable smooth surface $F \subset B^4$ without sphere components bounded by K. Considering the knot K as the branch locus for a branched double cover of S^3, and the surface F as a branch locus for a branched double cover of B^4, we can consider the orbifold Euler characteristic $\chi^{orb}(F) = \frac{1}{2}\chi(F)$ so that $\chi_4^{orb}(K)$ is the maximum of $\chi^{orb}(F)$ among such surfaces F. The division by 2 in the orbifold Euler characteristic is due to the degree of the cover. Now our initial inequalities takes the form:

$$(1) \qquad \frac{1}{2}|\sigma(K)| \leq \frac{1}{2} - \chi_4^{orb}(K) \leq u(K).$$

The *Gordian distance* $d(K_1, K_2)$ between two knots is the minimal number of crossing changes needed to convert K_1 into K_2. Essentially the same proof as for Inequality (1) shows that

$$(2) \qquad \frac{1}{2}|\sigma(K_1) - \sigma(K_2)| \leq \frac{1}{2} - \chi_4^{orb}(K_1 \# - mK_2) \leq d(K_1, K_2).$$

Indeed, $u(K)$ is just the Gordian distance of K with the unknot.

Both inequalities (1) and (2) extend to the setting of totally oriented Klein graphs, which are trivalent spatial graphs with extra coloring and orientation information. The simplest Klein graph is the θ-curve with edge colorings. In a sequence of crossing changes of such a graph it is sensible to keep track of whether a crossing change occurs between distinct edges, a mixed-color crossing change, or between an edge and itself, a same-color crossing change. For a sequence of crossing changes, we let m denote the number of mixed-color crossing changes and s the number of same-color crossing changes. The *Klein unknotting number* $u_{\lambda}(\theta)$ is the minimum of $m/2 + s$ over all crossing change sequences converting θ to the unknot. We prove:

Corollary 1.1. *If θ is a theta curve in S^3, then*

$$\frac{1}{4}|\sigma(\theta)| \leq \frac{1}{4} - \chi_4^{orb}(\theta) \leq u_{\curlywedge}(\theta).$$

The statement for theta curves is a corollary of Theorem 4.1.

Recently, the quest to understand unknotting numbers and sequences of crossing changes for θ-curves has been provoked by potential biological applications [BO18, OSB18, BBM+22]. In Section 5, we give examples demonstrating that the bounds of Corollary 1.1 improve previous bounds.

1.1. **Totally orientable Klein graphs.** More generally, we develop analogous bounds for the class of *totally oriented Klein graphs*.

A *spatial graph* is a smooth embedding in S^3 of a compact graph, allowing S^1 components, considered up to smooth isotopy. A trivalent spatial graph is a *web*. Without the embedding into S^3, a web is also known as a *cubic graph*. A *Klein coloring* or *Tait coloring* of a web is an assignment of a color r, g, b to each edge or loop so that edges of all three colors are incident to each vertex of the web. A web equipped with a Klein coloring is a *Klein graph*. Klein colorings arise naturally when considering quotients of the Klein group acting on 3-manifolds.

Given a Klein graph Γ, each unordered pair $\{i, j\} \subset \{r, g, b\}$ defines the *bicolored link* Γ_{ij} comprised of only the edges and loops colored i or j. A Klein graph is *3-Hamiltonian* if each bicolored link is actually a knot; signatures for 3-Hamiltonian Klein graphs were developed by Gille and Robert [GR18]. In the process of extending their work to relate signature, unknotting number, and χ_4^{orb}, we found that it was possible to drop the 3-Hamiltonian condition.

This enables an analogue to the classical extension of Inequality (1) to links. Effectively we obtain that for two Klein graphs related by crossing changes, the difference in their signatures gives a lower bound on an Euler characteristic of a certain type of cobordism between them. This, in turn, gives a lower bound on the Klein Gordian distance between them, though with adjustments based on easily calculable invariants of the two graphs.

Theorem 4.1. *Suppose that $\overrightarrow{\Gamma_1}$ and $\overrightarrow{\Gamma_2}$ are totally oriented Klein graphs that are related by a sequence of crossing changes. Set $V := |V(\Gamma_1)| = |V(\Gamma_2)|$ and $\mu := \mu(\Gamma_1) = \mu(\Gamma_2)$. Then:*

$$|\sigma(\overrightarrow{\Gamma_1}) - \sigma(\overrightarrow{\Gamma_2})| - \beta(\overrightarrow{\Gamma_1}) - \beta(\overrightarrow{\Gamma_2}) + 4\mu - 12 \leq 5 - 4\chi_4^{orb}(\overrightarrow{\Gamma_1}\#_3 - m\overrightarrow{\Gamma_2}) - 2V$$

$$\leq -4\chi_4^{orb}(\overrightarrow{\Gamma_1}, \overrightarrow{\Gamma_2}; s) - 2V$$

$$\leq 4d_{\curlywedge}(\overrightarrow{\Gamma_1}, \overrightarrow{\Gamma_2})$$

where $\overrightarrow{\Gamma_1}\#_3 - m\overrightarrow{\Gamma_2}$ is any vertex connected sum of $\overrightarrow{\Gamma_1}$ and $-m\overrightarrow{\Gamma_2}$ compatible with the total orientations.

Here β is the nullity, μ is the sum of the number of components of the bicolored links, and $d_{\curlywedge}$ is the Klein Gordian distance (akin to the Klein unknotting number). To obtain Theorem 4.1 we must impose the extra structure of a *total orientation* on our Klein graphs, which is a choice of orientation on each bicolored link. We denote a totally oriented Klein graph by $\overrightarrow{\Gamma}$.

The additional structure of a total orientation enables us to define in Section 2.4 for any totally oriented Klein graph $\overrightarrow{\Gamma}$ signatures $\zeta(\Gamma)$ and $\sigma(\overrightarrow{\Gamma})$. These are analogous to the Murasugi signature of an unoriented link and the signature of an oriented link, respectively, and extending [GR18]. The signature $\sigma(\overrightarrow{\Gamma})$ derives from a $\mathbb{Z}_2 \times \mathbb{Z}_2$ covering of B^4 branched over a properly embedded colored 2-complex F in B^4 called a *Klein foam* for which $\Gamma = \partial F$ in $S^3 = \partial B^4$. (See Definition 2.6 for the specifics of the foam.) The terms $\chi_4^{orb}(\overrightarrow{\Gamma_1}\#_3 - m\overrightarrow{\Gamma_2})$ and $\chi_4^{orb}(\overrightarrow{\Gamma_1}, \overrightarrow{\Gamma_2}; s)$ appearing in Theorem 4.1 both refer to *slice orbifold characteristics* of two types of totally oriented Klein foams relating $\overrightarrow{\Gamma_1}$ and $\overrightarrow{\Gamma_2}$. Here, $\chi_4^{orb}(\overrightarrow{\Gamma_1}\#_3 - m\overrightarrow{\Gamma_2})$ is a maximum taken over all of orientable slice foams bounding the vertex sum $\overrightarrow{\Gamma_1}\#_3 - m\overrightarrow{\Gamma_2}$, whereas $\chi_4^{orb}(\overrightarrow{\Gamma_1}, \overrightarrow{\Gamma_2}; s)$ calculates the *seamed cobordism characteristic* of a restricted type of foam where the singular set is a 1-manifold of which each edge runs from $\overrightarrow{\Gamma_1}$ to $\overrightarrow{\Gamma_2}$ (see Definitions 2.8 and 2.13).

In general the singular set of a Klein foam could have interior vertices. A total orientation on a Klein foam bestows an orientation on these vertices. A signed count of these vertices, the *signed seam vertex count*

sv, can be determined from the totally oriented Klein graph that is the boundary of the foam. This is an invariant of totally oriented Klein graphs, as we discuss in Section 3.2.

Theorem 4.3. *For a totally oriented Klein graph $\overrightarrow{\Gamma} \subset S^3$, we have*

$$|\sigma(\overrightarrow{\Gamma})| \leq 3 - |V(\Gamma)| + 2|\mathrm{sv}(\overrightarrow{\Gamma})| - 4\chi_4^{orb}(\overrightarrow{\Gamma}) - 2(\mu(\Gamma) - 3) + \beta(\overrightarrow{\Gamma}).$$

In Section 4.2, we turn our attention to the construction of the totally oriented seamed foamy cobordisms that are required for the third inequality in Theorem 4.1. As it will turn out, such objects naturally arise by stacking cobordisms resulting from crossing changes of totally oriented Klein graphs.

Theorem 4.9. *Suppose there is a sequence of s same-colored crossing changes and m mixed-colored crossing changes between totally oriented Klein graphs $\overrightarrow{\Gamma_1}$ and $\overrightarrow{\Gamma_2}$. Then there is a totally orientable seamed foamy cobordism $\overrightarrow{F}$ between $\overrightarrow{\Gamma_1}$ and $\overrightarrow{\Gamma_2}$ without bicolored spheres with*

$$\chi^{orb}(\overrightarrow{F}) = -|V(\overrightarrow{\Gamma_1})|/2 - (s + m/2).$$

Consequently, the seamed cobordism characteristic provides a lower bound on Klein Gordian distance, providing the last inequality in Theorem 4.1:

Corollary 1.2. *If the Klein Gordian distance between two Klein graphs Γ_1 and Γ_2 is $d_\curlywedge(\Gamma_1, \Gamma_2)$ then*

$$-\chi_4^{orb}(\Gamma_1, \Gamma_2; s) \leq d_\curlywedge(\Gamma_1, \Gamma_2) + \frac{1}{2}|V(\Gamma_1)|.$$

1.2. **Organization.** Section 2 collects all major definitions, including Klein graphs and unknotting numbers in 2.1, orientations and foams in 2.2, orbifold Euler characteristics in 2.3, and signatures in 2.4. In Section 3, we study the structure of total orientatons on Klein graphs, addressing foams with seam vertices in 3.1, orientations in 3.2, and cobordisms in 3.3. In Section 4 we prove our main results: Theorem 4.1, Theorem 4.3 and Theorem 4.9. In Section 5 we present several examples; in 5.1 we give improved bounds on unknotting number for a family of theta curves, and in 5.2 we compare the bounds of Theorem 4.1 with the notion of *strong* signature. Lastly, an appendix aggregates the behavior of various invariants under reversal, mirroring and connected sum for convenience.

2. DEFINITIONS

2.1. **Klein graphs and unknotting numbers.**

Definition 2.1 (Klein graphs). A *Klein graph* Γ is a finite trivalent graph properly embedded in S^3 with a 3-coloring: each edge is colored either red, blue, green, each vertex of the graph is incident to edges of all three colors, and the graph contains all three colors. We denote red, blue, green by r, b, g, respectively, and often write i, j, k for arbitrary distinct colors in $\{r, g, b\}$. The name arises from the fact that for a given Klein graph there is a regular orbifold cover of S^3 with branch set the Klein graph and with deck group the Klein group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. A Klein graph is *3-Hamiltonian* if for any two distinct colors the union of all edges of those colors is a single cycle. (Observe that a 3-Hamiltonian Klein graph is connected.) Klein graphs may have 'knot components,' i.e. components without vertices. We will assume throughout that Klein graphs are smoothly embedded (except at vertices) and work with surfaces up to diffeomorphism. Throughout, we let $|V(\Gamma)|$ denote the number of vertices of Γ. The number of edges of Γ is equal to $3|V(\Gamma)|/2$ and so the Euler characteristic $\chi(\Gamma) = -|V(\Gamma)|/2$.

Definition 2.2 (Bicolored links, component count). Suppose Γ is a Klein graph. For each color i, let Γ_i be the union of the edges colored i. For each pair of distinct colors i, j, the *bicolored link* Γ_{ij} is (the closure of) the union $\Gamma_i \cup \Gamma_j$. Observe that Γ is 3-Hamiltonian if and only if each bicolored link is a knot.

Define the *component count* $\mu(L)$ of a link L to be its number of components and the *component count* $\mu(\Gamma)$ of a Klein graph Γ to be the total number of components of its bicolored links. Hence $\mu(\Gamma) = \mu(\Gamma_{rb}) + \mu(\Gamma_{bg}) + \mu(\Gamma_{rg})$.

Definition 2.3 (Crossing changes). A crossing change on a Klein graph can occur either between edges of the same color or between two distinct colors. We call these types of crossing changes respectively a *same-color crossing change* and a *mixed-color crossing change*, or more simply a *same crossing change* and a *mixed crossing change*. For a sequence of crossing changes on a Klein graph, we let s denote the number of

same crossing changes and m the number of mixed crossing changes. The *Klein Gordian distance* $d_\curlywedge(\Gamma_1, \Gamma_2)$ between two Klein graphs is the minimum of $s + m/2$ over all sequences of crossing changes converting Γ_1 into Γ_2, or ∞ if there is no such sequence.

The *Klein unknotting number* $u_\curlywedge(\Gamma)$ of a Klein graph Γ is $d_\curlywedge(\Gamma, U)$ where U is a Klein graph embedded in a 2-sphere in S^3. Of course, the number $u_\curlywedge(\Gamma)$ is finite if and only if Γ is abstractly planar. (If Γ is abstractly planar, then by [Mas69], up to isotopy in S^3, there is a unique such U.)

Definition 2.4 (Connected sums). Let Γ_1 and Γ_2 be two Klein graphs in distinct copies of S^3.

An *order 2 connected sum* or *edge connected sum* $\Gamma_1 \#_2 \Gamma_2$ is obtained as follows. Choose a point in the interior an edge of each graph of the same color and delete the interior of a closed ball about each point meeting its graph in an arc in the interior of its edge. Glue together the remnants by an orientation reversing homeomorphism of the resulting two S^2 boundary components so that the pairs of points of Γ_1 and Γ_2 in those spheres match up. The result $\Gamma_1 \#_2 \Gamma_2$ is again a Klein graph. Note that the result depends on which edges contain the points and how the pairs of points in the identified spheres match up.

An *order 3 connected sum* or *vertex connected sum* $\Gamma_1 \#_3 \Gamma_2$ is obtained similarly: Here we choose a vertex of each graph and delete the interior of a closed ball about each of these vertices meeting its graph in three proper subarcs of the edges meeting that vertex. Now glue together the remnants by an orientation reversing homeomorphism of the resulting two S^2 boundary components so that the triples of points of Γ_1 and Γ_2 in those spheres match up according to their colors. The result $\Gamma_1 \#_3 \Gamma_2$ is again a Klein graph. Note that the result depends only on which vertices are chosen [Wol87].

In Section 2.2 we introduce a notion of *total orientation* on a Klein graph. When two totally oriented Klein graphs have total orientations that match on a pair of edges or are opposite on a pair of vertices, these connected sum operations may be performed along the pair in a manner respecting the total orientations in the obvious way to confer a total orientation on the resulting sum.

2.2. **Orientations and Foams.** Here we give a first pass at defining a kind of orientation for Klein graphs and foams that will enable an Euler characteristic count in Definition 2.13 that is analogous to the slice genus of an oriented link. In Section 3 we more fully develop this orientation and its implications.

Definition 2.5 (Klein graph: Total orientation, double orientation). A *total orientation* on a Klein graph Γ is a choice of orientation on each of its bicolored links. A total orientation on Γ induces a *double orientation* on each of its edges; that is, an i-colored edge inherits an ij-orientation from the orientation on Γ_{ij} and an ik-orientation from the orientation on Γ_{ik}. If the ij- and ik-orientations on an i-edge are coherent, we say the doubly oriented edge has a *parallel* double orientation and its sign is $+$. Otherwise the doubly oriented edge has an *antiparallel* double orientation and its sign is $-$.

We denote mirroring and reversal by $m(\Gamma)$ and $-\Gamma$, respectively, and for emphasis, sometimes indicate totally oriented objects with an arrow, *e.g.* $(\overrightarrow{\Gamma})$.

Definition 2.6 (Foams). A *foam (with boundary)* is a properly embedded smooth two-dimensional CW-complex F in a four manifold (here B^4 or $S^3 \times I$) in which every point in the interior or the boundary is a regular point or a singular point, meaning it has a neighborhood diffeomorphic to one of the pictures in Figure 1. The singular points $s(F)$ form a 1–complex, called the *seam points*. Interior vertices of $s(F)$ are called *seam vertices*. (The boundary of $s(F)$ is the collection of univalent vertices of $s(F)$, and these are the vertices of the trivalent graph ∂F.) The *facets* of a foam are the connected components of $F - s(F)$, consisting of sets of regular points, and are smooth surfaces.

A *Klein foam (with boundary)* is a foam with a 3-coloring whose boundary is a union of Klein graphs. That is, each facet of the foam is colored either red, blue, green and each edge of $s(F)$ is incident to facets of all three colors. In a Klein foam with boundary the vertices of the bounding Klein graphs are the end points of $s(F)$, and the edges are part of $F - s(F)$.

Definition 2.7 (Klein foam: Total orientation, double orientation). A Klein foam F is *totally orientable* (in [KR21], this is termed *admissible*) if each of its bicolored surfaces are orientable. As with a Klein graph Γ and its edges, a *total orientation* on a totally orientable F is a choice of orientation on each bicolored surface, and this induces a *double orientation* on each of its facets.

Further observe that a total orientation on a Klein foam F with boundary $\Gamma = \partial F$ induces a total orientation on the Klein graph Γ.
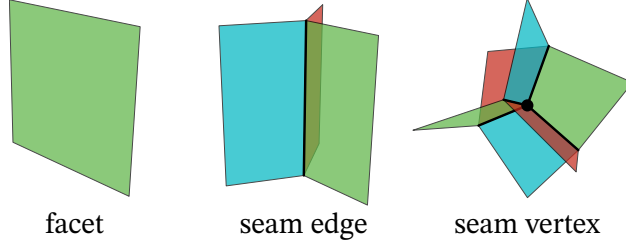
FIGURE 1. Neighborhoods of a point in a facet, a point in a seam edge, and a seam vertex.

Definition 2.8 (Foamy cobordisms)**.** Let F be a Klein foam that is properly embedded in $S^3 \times [0,1]$. Let $\partial F = \Gamma_0 \cup -m\Gamma_1$, where $\Gamma_0 \subset S^3 \times \{0\}$ and $\Gamma_1 \subset S^3 \times \{1\}$ are each Klein graphs or the empty set. We call such a foam a *foamy cobordism*. Note that $s(F)$ is a 1-manifold with boundary if and only if F has no seam vertices. Three particular instances of a foamy cobordism are

- a *slice foam* for Γ_0, in which Γ_1 is the empty set but Γ_0 is not,
- a *spanning foam* for Γ_0, which is a slice foam without seam vertices, and
- a *seamed foamy cobordism* from Γ_0 to Γ_1, which is a foamy cobordism without seam vertices in which each edge component of $s(F)$ has one endpoint lying on each of Γ_0 and Γ_1; closed components of $s(F)$ are permitted.

By attaching a 4-ball to $S^3 \times \{1\}$, spanning foams and slice foams may be regarded as being properly embedded in B^4.

Definition 2.9 (Bicolored surfaces)**.** Suppose F is a Klein foam. For each color i, let F_i be the union of the facets colored i. For each pair of distinct colors i, j, the *bicolored surface* F_{ij} is the closure of $F_i \cup F_j$. Observe that F_{ij} is indeed a compact surface and contains $s(F)$, but may be disconnected. A *bicolored sphere* of a Klein foam F is a spherical component of some bicolored surface. When a Klein foam is connected, any closed component of a bicolored surface necessarily contains both colors. Note that a monocolored sphere (a sphere component of a single color) is a bicolored sphere by definition.

Definition 2.10 (The quantity $|\mathrm{sv}|$)**.** For a Klein graph Γ in S^3, let $|\mathrm{sv}|(\Gamma)$ be the minimum number of seam vertices among the slice foams bounded by Γ. If $\vec{\Gamma}$ is Γ with a total orientation, let $|\mathrm{sv}|(\vec{\Gamma})$ be the minimum number of seam vertices among the totally oriented slicing foams bounded by $\vec{\Gamma}$.

Similarly, for a pair of Klein graphs Γ_0 and Γ_1, let $|\mathrm{sv}|(\Gamma_0, \Gamma_1)$ be the minimum number of seam vertices among the foamy cobordisms from Γ_0 to Γ_1. Analogously define $|\mathrm{sv}|(\vec{\Gamma_0}, \vec{\Gamma_1})$ when the two graphs are totally oriented.

Remark 2.11. In fact, as we will see in Section 3, seam vertices of totally oriented foams have two isomorphism types. This observation with a choice which to call 'positive' lifts $|\mathrm{sv}|$ to a $\mathbb{Z}$ valued function sv on totally oriented Klein graphs so that the absolute value of $\mathrm{sv}(\vec{\Gamma})$ is $|\mathrm{sv}|(\vec{\Gamma})$. In Lemma 3.12, we show that $\mathrm{sv}(\vec{\Gamma})$ is easily determined from the combinatorics of the totally oriented Klein graph.

Lemma 2.12. *Given a totally orientable Klein foam F, there exists a totally orientable Klein foam F' without bicolored spheres such that $\partial F = \partial F'$.*

Proof. Let F be a totally orientable Klein foam. Since F is compact, it has at most finitely many bicolored spheres. For each bicolored sphere of F, choose a facet of F in that sphere and connect sum a torus onto its interior. The total orientation of F extends across these connected sums. Thus the resulting foam F' continues to be totally orientable with $\partial F = \partial F'$ yet has no bicolored spheres. □

2.3. **Orbifold Euler Characteristics.**

Definition 2.13.

- For a Klein foam F, its *orbifold Euler characteristic* is

$$\chi^{orb}(F) = \frac{1}{2}\chi(F) - \frac{1}{4}\chi(s(F)).$$

5

- For a totally oriented Klein graph $\vec{\Gamma}$, its *slice orbifold Euler characteristic* is

$$\chi_4^{orb}(\vec{\Gamma}) = \max \chi^{orb}(F)$$

where the maximum is taken over all totally oriented slice foams F in B^4 with $\vec{\Gamma} = \partial F$ having $|\mathrm{sv}|(\vec{\Gamma})$ seam vertices and no bicolored spheres.
- For a Klein graph Γ (without a specified total orientation), its *slice orbifold Euler characteristic* is

$$\chi_4^{orb}(\Gamma) = \max \chi^{orb}(F)$$

where the maximum is taken over all totally orientable slice foams F in B^4 with $\Gamma = \partial F$ having $|\mathrm{sv}|(\partial F)$ seam vertices and no bicolored spheres.
- For two Klein graphs Γ_1, Γ_2 that cobound a totally orientable seamed foamy cobordism, the *seamed cobordism characteristic* is

$$\chi_4^{orb}(\Gamma_1, \Gamma_2; s) = \max \chi^{orb}(G)$$

where the maximum is taken over all totally orientable seamed foamy cobordisms G from Γ_1 to Γ_2 having no bicolored spheres.

Lemma 2.12 gives an easy way to eliminate bicolored spheres. Lemmas 2.17 and 2.18 show that these maxima are bounded above.

Remark 2.14. As we observe in Lemmas 3.1 and 3.3, a Klein graph need not bound a totally orientable slice foam and there need not be a seamed foamy cobordism between two Klein graphs with the same number of vertices. Nevertheless, as shown in Theorem 3.6, these always exist if we permit the foams to have seam vertices.

Remark 2.15. (1) Definition 2.13 agrees with standard definitions of the orbifold Euler characteristic for an orbifold Q,

$$\chi(Q) = \sum_{q \in Q} \frac{(-1)^{\dim q}}{|G(q)|},$$

where $G(q)$ is the local group at the open cell q and the sum is over all open cells [CHK00]. In our case, the local group of open 2-dimensional cells of F and the open edges of Γ is $\mathbb{Z}/2\mathbb{Z}$, acting by the involution of the double cover. The local group of the seams (including seam vertices) is the entire Klein group.
  (2) Permitting arbitrarily many bicolored spheres allows one to construct totally orientable foams having the same boundary yet with arbitrarily large χ_4^{orb} as follows. Let F be any totally orientable Klein foam and let F_0 be the boundaryless Klein foam in S^4 that is a suspension of a theta curve. Letting F' be a connected sum of F and F_0 along points in the interiors of a facets of F and F_0 of the same color, as shown in Figure 2(Left), we see that $\chi_4^{orb}(F') = \chi_4^{orb}(F) + \frac{1}{2}$.
  (3) Permitting arbitrarily many seam vertices similarly allows one to construct totally orientable foams having the same totally oriented boundary yet with arbitrarily large χ_4^{orb}. Let F be any totally orientable Klein foam and let F_0 be the boundaryless Klein foam in S^4 that is the suspension of the tetrahedral graph. Note that F_0 has two seam vertices. Letting F' be the connected sum of F and F_0 along points in the interior of seam edges of F and F_0, as shown in Figure 2(Right), we see that $\chi_4^{orb}(F') = \chi_4^{orb}(F) + 1$. Also note that if F has no bicolored spheres, then F' doesn't either.

Lemma 2.16. *For a foam F without seam vertices, we have*

$$4\chi^{orb}(F) = (\chi(F_{rb}) + \chi(F_{bg}) + \chi(F_{rg})) - |V(\partial F)|$$

Proof. Observe that the sum $\chi(F_{rb}) + \chi(F_{bg}) + \chi(F_{rg})$ counts the Euler characteristic of $s(F)$ three times but of $F - s(F)$ only twice. Since there are no seam vertices, each component of $s(F)$ is either a circle or an edge with its endpoints as vertices of the Klein graph ∂F. Hence $\chi(s(F)) = \frac{1}{2}|V(\partial F)|$. Now, beginning
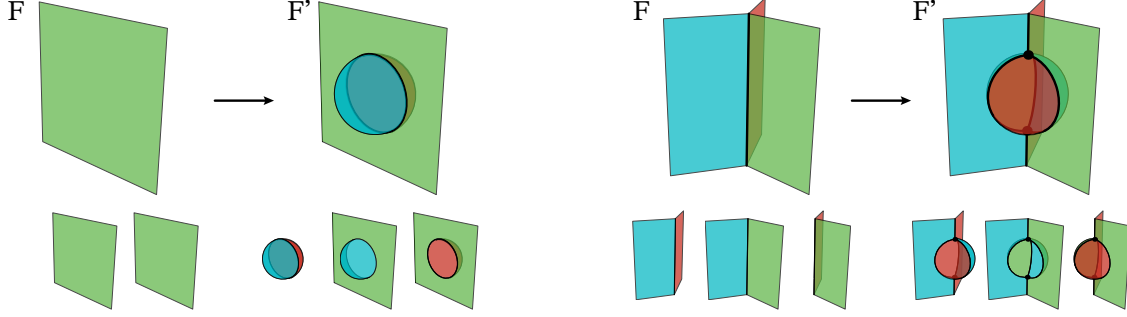
FIGURE 2. (Left) A portion of a facet of F and then the result F' of its connected sum with the suspension of the theta graph. (Right) A neighborhood of a point in a seam edge of F and then the result F' of its connected sum with the suspension of the tetrahedral graph.

with Definition 2.13, we may count

$$4\chi^{orb}(F) = 2\chi(F) - \chi(s(F))$$
$$= 2\chi(F - s(F)) + \chi(s(F))$$
$$= (2\chi(F - s(F)) + 3\chi(s(F))) - 2\chi(s(F))$$
$$= (\chi(F_{rb}) + \chi(F_{bg}) + \chi(F_{rg})) - |V(\partial F)|.$$

$\square$

The following two lemmas establish that $\chi^{orb}_4(F)$ and $\chi^{orb}_4(\Gamma_0, \Gamma_1; s)$ are bounded above and thus $\chi^{orb}_4(\Gamma)$ is well-defined.

Lemma 2.17. *Suppose Γ is a Klein graph that bounds a totally orientable slice foam. Then for any totally orientable slice foam F without bicolored spheres such that $\Gamma = \partial F$, $4\chi^{orb}(F) \leq \mu(\Gamma) - |V(\Gamma)|$. Consequently, $4\chi^{orb}_4(\Gamma) \leq \mu(\Gamma) - |V(\Gamma)|$.*
Furthermore, if Γ has no knot components then $4\chi^{orb}_4(\Gamma) \leq \frac{1}{2}|V(\Gamma)|$.

Proof. Let F be a totally orientable slice foam without bicolored spheres for the Klein graph Γ.
By Lemma 2.16,

$$4\chi^{orb}(F) = (\chi(F_{rb}) + \chi(F_{bg}) + \chi(F_{rg})) - |V(\partial F)|.$$

Observe that since $\chi(F_{ij})$ is the sum of the Euler characteristics of the components of F_{ij} which are orientable surfaces and no component is a sphere (because F has no bicolored spheres), we have $\chi(F_{ij}) \leq |\Gamma_{ij}|$ with equality realized exactly when F_{ij} is a union of disks and tori. Thus $\chi(F_{rg}) + \chi(F_{bg}) + \chi(F_{rg}) \leq \mu(\Gamma)$. Hence $4\chi^{orb}_4(F) \leq \mu(\Gamma) - |V(\Gamma)|$. Therefore $4\chi^{orb}_4(\Gamma) \leq \mu(\Gamma) - |V(\Gamma)|$.
If Γ has no knot components, then since each component of a bicolored link of Γ must involve at least two vertice, we have $3|V(\Gamma)| \geq 2\mu(\Gamma)$. Hence $\mu(\Gamma) - |V(\Gamma)| \leq \frac{3}{2}|V(\Gamma)| - |V(\Gamma)| = \frac{1}{2}|V(\Gamma)|$. Thus we have $4\chi^{orb}_4(F) \leq \frac{1}{2}|V(\Gamma)|$. $\square$

Lemma 2.18. *Suppose Γ_0 and Γ_1 are Klein graphs each with V vertices and F is a totally orientable seamed foamy cobordism without bicolored spheres between them. Then $4\chi^{orb}(F) \leq \mu(\Gamma_0) + \mu(\Gamma_1) - 2V$. Consequently, $4\chi^{orb}_4(\Gamma_0, \Gamma_1; s) \leq \mu(\Gamma_0) + \mu(\Gamma_1) - 2V$.*

Proof. Viewing the cobordism as $F \subset S^3 \times [0,1]$, we may choose a properly embedded arc in $S^3 \times [0,1]$ with endpoints in distinct boundary components that is disjoint from F. The exterior of such an arc is B^4 with a proper embedding of F having boundary as the split union $\Gamma_0 \sqcup -m\Gamma_1$ in S^3. From Lemma 2.17, it then immediately follows that $4\chi^{orb}(F) \leq \mu(\Gamma_0) + \mu(\Gamma_1) - 2V$. Consequently, $4\chi^{orb}_4(\Gamma_0, \Gamma_1; s) \leq \mu(\Gamma_0) + \mu(\Gamma_1) - 2V$. $\square$

2.4. Signatures. The *weak Euler number $e(F)$* of a Klein foam is the sum of the relative normal Euler numbers of the bicolored surfaces, $e(F) := e(F_{rb}) + e(F_{bg}) + e(F_{rg})$. In [GR18, Definition 3.10], Gille-Robert define the signature of a 3-Hamiltonian Klein graph $\Gamma \subset S^3$ to be

$$\sigma(\Gamma) = \sigma(W_F) + \frac{1}{2}e(F)$$

where F is a properly embedded spanning Klein foam for Γ in B^4, $e(F)$ is its weak Euler number, W_F is the 4-manifold Klein cover of B^4 branched over F, and $\sigma(W_F)$ is its signature. They then prove the following:

Theorem 2.19 ([GR18, Proposition 3.14]). *Suppose that Γ is a 3-Hamiltonian Klein graph with bicolored knots Γ_{ij} where $\{i, j, k\} = \{r, g, b\}$. Then*

$$\sigma(\Gamma) = \sigma(\Gamma_{rb}) + \sigma(\Gamma_{bg}) + \sigma(\Gamma_{rg})$$

As suggested by the 2nd footnote on page 2 of [GR18], their definition of the signature of a 3-Hamiltonian Klein graph and Theorem 2.19 can be extended to Klein graphs that are not 3-Hamiltonian. This echos the extension of signatures of knots to links where orientation considerations are important. Let us review some of these details.

Let $\vec{L}$ be a link L endowed with an orientation. Given any oriented Seifert surface F for $\vec{L}$, the signature $\sigma(\vec{L})$ of $\vec{L}$ is the signature of the symmetrized Seifert pairing $V + V^T$ of F. Equivalently, after isotoping F rel-∂ to be properly embedded in B^4, $\sigma(\vec{L}) = \sigma(M_F)$, the signature of the double of B^4 branched over F.

Definition 2.20 (Link nullity). The *nullity of the link $\beta(L)$* is the nullity of $(V + V^T)$ plus $r - 1$, where r is the number of connected components of F, as in [Pow17]. Note that the nullity of a link is independent of choices of orientation on the link [KT76, Theorem 2.1]. Note that $0 \le \beta(\vec{L}) \le \mu(L) - 1$ for any oriented link $\vec{L}$, and in particular, $\mu(L) - 1 = 0$ and $\beta(L) = 0$ for a knot, where $\mu(L)$ is the number of components of the link. While nullity is originally defined as the nullity of $(V + V^T)$ plus r, see [Mur65] and [KT76, Remark 1.2], the version of [Pow17] makes it additive under connected sum.

Theorem 2.21 ([KT76, Theorem 2.2]). *Let L_1 and L_2 be links in S^3. Then:*

$$\beta(L_1 \sqcup L_2) = \beta(L_1) + \beta(L_2) + 1 \quad and \quad \beta(L_1 \# L_2) = \beta(L_1) + \beta(L_2)$$

Murasugi [Mur65] defines the *total linking number $\lambda(\vec{L})$* of $\vec{L}$ to be the sum of the linking numbers of all unordered pairs of components of $\vec{L}$ and then shows that the sum $\sigma(\vec{L}) + \lambda(\vec{L})$ is independent of the choice of orientation on L. Hence he defines the signature of the unoriented link L to be $\zeta(L) = \sigma(\vec{L}) + \lambda(\vec{L})$. (See also Definition 3.12 and Remark 3.13 of [GR18].) Gordon-Litherland show that if F is any spanning surface for L, then $\zeta(L) = \sigma(M_F) + e(F)$ [GL78, Corollary 5']. Hence, equipping L with an orientation as $\vec{L}$, we have $\sigma(\vec{L}) = \sigma(M_F) + e(F) - \lambda(\vec{L})$ regardless of any orientation on F.

The following theorem is suggested but not explicitly stated in [GR18, Footnote 2].

Theorem 2.22. *Let Γ be a Klein graph in S^3. Let F be a spanning foam for Γ in B^4. Then*
- *the integer $\zeta(\Gamma) := \sigma(W_F) + \frac{1}{2}e(F)$ depends only on Γ, and*
- $\zeta(\Gamma) = \zeta(\Gamma_{rb}) + \zeta(\Gamma_{bg}) + \zeta(\Gamma_{rg})$.

We say $\zeta(\Gamma)$ is the signature of Γ.

Proof. The first part, that the sum $\sigma(W_F) + \frac{1}{2}e(F)$ is independent of the choice of spanning foam F for Γ, follows exactly as in the proof of [GR18, Theorem 3.11] (skipping the middle part). There [GR18, Proposition 3.7] continues to apply because $W = B^4$ in this situation.

The second part follows the first half of the proof of [GR18, Theorem 3.14] where [GL78, Theorem 2] extends directly to links as discussed in [GL78, §5] while [GL78, Theorem 5'] is used in the place of [GL78, Theorem 5]. □

Definition 2.23 (Total linking number, and nullity). Let $\vec{\Gamma}$ be a totally oriented Klein graph. Then the *total linking number* of $\vec{\Gamma}$ is the sum of the total linking numbers of its bicolored links. That is,

$$\lambda(\vec{\Gamma}) = \lambda(\vec{\Gamma}_{rb}) + \lambda(\vec{\Gamma}_{bg}) + \lambda(\vec{\Gamma}_{rg}).$$

In accordance with the relation $\zeta(L) = \sigma(\vec{L}) + \lambda(\vec{L})$ between the Murasugi signature of an unoriented link L and the signature and total linking number, the *signature* of $\vec{\Gamma}$ is then $\sigma(\vec{\Gamma}) = \zeta(\Gamma) - \lambda(\vec{\Gamma})$. Similarly, define the *nullity* $\beta(\Gamma) = \beta(\Gamma_{rb}) + \beta(\Gamma_{bg}) + \beta(\Gamma_{rg})$ to be the sum of the nullities of the bicolored links.

It follows immediately from these definitions that the operations of mirroring and total orientation reversal of a totally oriented Klein graph leave μ the component count and β unchanged while λ is negated by mirroring but unchanged by total orientation reversal. This is recorded in the following lemma.

Lemma 2.24. *Let $\vec{\Gamma}$ be a totally oriented Klein graph. Then*

$$
\begin{array}{rclcll}
\lambda(-\vec{\Gamma}) & = & \lambda(\vec{\Gamma}), & & & and \\
\lambda(m(\vec{\Gamma})) & = & -\lambda(\vec{\Gamma}), & & & while \\
\mu(-\vec{\Gamma}) & = & \mu(m(\vec{\Gamma})) & = & \mu(\vec{\Gamma}), & and \\
\beta(-\vec{\Gamma}) & = & \beta(m(\vec{\Gamma})) & = & \beta(\vec{\Gamma}). &
\end{array}
$$

Lemma 2.25. *Let Γ_1 and Γ_2 be Klein graphs. Then we have*

$$
\beta(\Gamma_1 \#_2 \Gamma_2) = \beta(\Gamma_1) + \beta(\Gamma_2) + 1 \quad and \quad \beta(\Gamma_1 \#_3 \Gamma_2) = \beta(\Gamma_1) + \beta(\Gamma_2)
$$

as well as

$$
\mu(\Gamma_1 \#_2 \Gamma_2) = \mu(\Gamma_1) + \mu(\Gamma_2) - 2 \quad and \quad \mu(\Gamma_1 \#_3 \Gamma_2) = \mu(\Gamma_1) + \mu(\Gamma_2) - 3
$$

Proof. The statement for the nullity β follows from the fact that, as defined, the nullity of a Klein graph is the sum of the nullities of its bicolored links. Note that an order 2 sum, say along a red edge, causes a split union of the two blue-green links. The statement for the component count μ is a straightforward count minding how many bicolored link components are merged under a sum. $\square$

Remark 2.26. Observe that for a 3-Hamiltonian graph $\vec{\Gamma}$ we have $\lambda(\vec{\Gamma}) = 0$, $\beta(\vec{\Gamma}) = 0$, and $\mu(\vec{\Gamma}) = 3$. Moreover, $\sigma(\vec{\Gamma})$ is independent of the total orientation.

Armed with Definition 2.23, Theorem 2.22 immediately allows for the calculation of the signature of a totally oriented Klein graph as a sum of the signatures of its oriented bicolored links.

Corollary 2.27. *Let $\vec{\Gamma}$ be a totally oriented Klein graph. Then*

$$
\sigma(\vec{\Gamma}) = \sigma(\vec{\Gamma}_{rb}) + \sigma(\vec{\Gamma}_{bg}) + \sigma(\vec{\Gamma}_{rg}).
$$

Proof.

$$
\begin{aligned}
\sigma(\vec{\Gamma}) &= \zeta(\Gamma) - \lambda(\vec{\Gamma}) \\
&= (\zeta(\Gamma_{rb}) + \zeta(\Gamma_{bg}) + \zeta(\Gamma_{rg})) - \left(\lambda(\vec{\Gamma}_{rb}) + \lambda(\vec{\Gamma}_{bg}) + \lambda(\vec{\Gamma}_{rg}) \right) \\
&= \left(\zeta(\Gamma_{rb}) - \lambda(\vec{\Gamma}_{rb}) \right) + \left(\zeta(\Gamma_{bg}) - \lambda(\vec{\Gamma}_{bg}) \right) + \left(\zeta(\Gamma_{rg}) - \lambda(\vec{\Gamma}_{rg}) \right) \\
&= \sigma(\vec{\Gamma}_{rb}) + \sigma(\vec{\Gamma}_{bg}) + \sigma(\vec{\Gamma}_{rg})
\end{aligned}
$$

$\square$

Remark 2.28. In [GR18, Proposition 3.14] (which we have only partially stated in Theorem 2.19) the hypothesis that the Klein graph Γ is 3-Hamiltonian is indeed used for the *strong signature* $\tilde{\sigma}(\Gamma)$; see Section 5.2. Their definition of this strong signature uses that the double branched cover of each bicolored knot Γ_{ij} is a rational homology sphere to ensure that the lift of the third color of edges Γ_k is a link of rationally null-homologous knots. The double branched cover of a knot is necessarily a rational homology sphere while the double branched cover of a link of more than one component might not be.

3. Total orientations of Klein graphs and seam vertices

3.1. **Foams with seam vertices.** For any spatial Klein graph Γ in S^3, Gille-Robert show how to construct a spanning foam (ie. a Klein foam without seam vertices) in B^4 whose boundary is Γ [GR18, Proposition 2.4]. However, some Klein graphs do not bound any totally orientable slice foam as we now observe.

Let $\Gamma_{tet} \subset S^3$ be the (planar) tetrahedral Klein graph shown in Figure 3. Define F_{vtx} to be the Klein foam that arises from the standard neighborhood (B^4, F_{vtx}) of a seam vertex of a Klein foam, a cone on (S^3, Γ_{tet}). Also note that Γ_{tet} is 3-Hamiltonian.
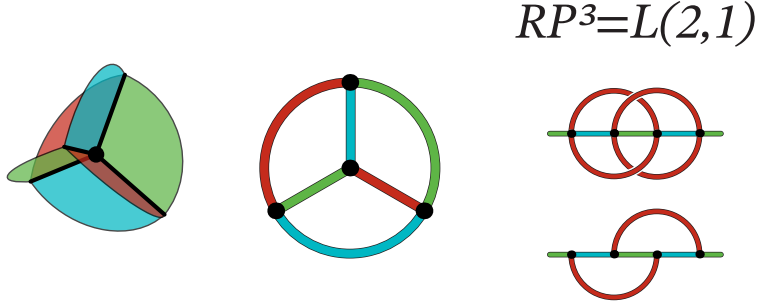
$RP^3=L(2,1)$

FIGURE 3. (Left) A neighborhood of a seam vertex. (Center) The planar (trivial) Klein tetrahedral graph is the boundary of a regular neighborhood of a seam vertex. (Right) The Klein cover of tetrahedral graph.

Lemma 3.1. *While F_{vtx} is totally orientable, $\Gamma_{tet} = \partial F_{vtx}$ does not bound a totally orientable Klein foam without seam vertices.*

Proof. Suppose F is a totally orientable Klein foam for Γ_{tet} without seam vertices. Since Γ_{tet} has just four vertices, F has exactly two seam edges. By symmetry, we may assume each seam edge has the same endpoints as a red edge. Then the two blue edges and two seam edges form a single cycle as do the two green edges and two seam edges. Thus the facets F_b and F_g are each connected with connected boundary.

Since F is totally orientable, the surface F_{bg} is orientable. Choose an orientation of F_{bg}. This induces an orientation on each F_b and F_g. Moreover, the bg-orientations on the blue and green edges of Γ_{tet} arising from ∂F_{bg} are also induced from the boundary orientations on ∂F_b and ∂F_g However, with these bg-orientations, one seam edge connects the tips of the blue edges while the other connects their tails. Yet this is inconsistent with any orientation induced from ∂F_b. $\square$

Remark 3.2. The Klein graph $\Gamma_{tet}\#_2\Gamma_{tet}$ bounds a totally orientable slice foam only for certain total orientations while other total orientations only bound totally oriented foams with at least two seam vertices.

Lemma 3.3. *Let $\Gamma_{2\theta} = \Gamma_\theta\#_2\Gamma_\theta$ be the edge connected sum of two trivial theta curves. There is no totally orientable seamed foamy cobordism from Γ_{tet} to $\Gamma_{2\theta}$, even though they have the same number of vertices.*

Proof. Suppose G were a totally orientable seamed foamy cobordism from Γ_{tet} to $\Gamma_{2\theta}$. Then a total orientation on G induces a total orientation on $\Gamma_{2\theta}$. Observe that $\Gamma_{2\theta}$ is the boundary of a 2-complex H in $\mathbb{R}^3 \subset S^3$ formed as the product of an interval with a planar graph in the form of the letter 'H'. With its interior pushed into B^4, H is a totally orientable slice foam for $\Gamma_{2\theta}$. Moreover, one readily observes that any total orientation on $\Gamma_{2\theta}$ arises as the boundary of some total orientation on H. Hence, with the total orientation on $\Gamma_{2\theta}$ induced from the total orientation on G, there is a total orientation on H so that $G\cup_{\Gamma_{2\theta}} H$ is a totally orientable slice foam for Γ_{tet}. However this contradicts Lemma 3.1. $\square$

In contrast to Lemmas 3.1 and 3.3, if we permit our foams to have seam vertices, we can always find totally orientable foams.

Example 3.4. The Klein cover of Γ_{tet} is $\mathbb{RP}^3$ and the Klein cover of F_{vtx} is the cone on $\mathbb{RP}^3$. In the double branched cover over the unknot $(\Gamma_{tet})_{ij}$, the edges $(\Gamma_{tet})_k$ lift to a Hopf link $(\widetilde{\Gamma}_{tet})_k$. Since the Klein cover of Γ_{tet} is then the double branched cover of $(\widetilde{\Gamma}_{tet})_k$, we see that the Klein cover of Γ_{tet} is $\mathbb{RP}^3$. Since (B^4, F_{vtx}) is a cone on (S^3, Γ_{tet}), this then implies that the Klein cover of F_{vtx} is the cone on $\mathbb{RP}^3$. See Figure 3.

Remark 3.5. Observe that Example 3.4 shows that the Klein cover of a foam with seam vertices is not a 4-manifold but rather a pseudo 4-manifold where each singularity is a cone on $\mathbb{RP}^3$.

Theorem 3.6. *Every totally oriented Klein graph is the boundary of a totally oriented Klein foam, possibly with seam vertices.*

Proof. Let $\vec{\Gamma}$ be a totally oriented Klein graph. We will induct on the number of vertices of $\vec{\Gamma}$.
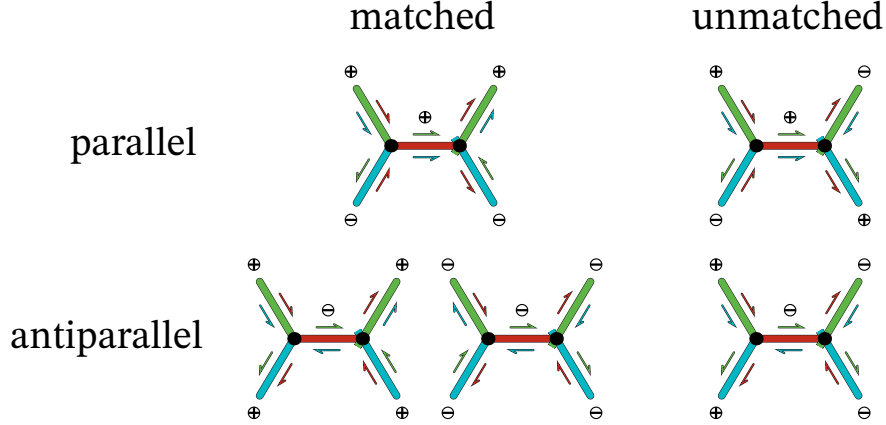
FIGURE 4. An edge of a totally oriented Klein graph is either parallel or antiparallel and either matched or unmatched. Up to symmetry, there are two possibilities for a matched antiparallel edge and one otherwise; these five possibilities are shown for a red edge.

If $\vec{\Gamma}$ has no vertices, then $\vec{\Gamma}$ is a link. First, by mixed crossing changes, $\vec{\Gamma}$ may be separated into a split link where each split component is a link of a single color. These mixed crossing changes may be realized by a sequence of foamy cobordisms F_m that respect the double orientations as done in the proof of Theorem 4.9; one only needs to be mindful of the ij-orientations on a mixed crossing change between an i-edge and a j-edge. Next, two linked components of the same color but with double orientations of opposite sign may be unlinked as follows. Use the elementary open/close cobordism as in Figure 6(Top Right) to 'slice open' one of the knot components. (This basically connect sums the knot with a trivial theta curve.) Then use mixed crossing changes with the two new edges to split away the other knot component. Thereafter the sliced open component can be sewn closed using the elementary open/close cobordism again but in the other direction.

Eventually, we have a split link where each split component is a (possibly multi-component) link of a single color and where the two orientations on each component are all parallel or all antiparallel. Each split component bounds a Seifert surface, and we may arrange that the Seifert surfaces for the split components are mutually disjoint. We color each Seifert surface according to the color of its bounding split component and we endow each with two orientations induced from the two orientations on the split component. The result is a foam, bounded by the split link, where the orientations of the bicolored surfaces are coherent, giving a total orientation to the foam. Together with the foams from the previous cobordisms, we obtain a totally oriented foamy cobordism from the totally oriented, Klein colored link $\vec{\Gamma}$ to the empty set. Thus $\vec{\Gamma}$ is the boundary of a totally oriented slice foam.

Now assume $\vec{\Gamma}$ has vertices. Note that a vertex has an odd number of negative edges incident to it (Definition 2.5). Since $\vec{\Gamma}$ has a vertex, it has an edge of each color.

Let e be a k-edge. Observe that the two i-edges adjacent to e have the same sign if and only if the two j-edges adjacent to e also have the same sign. Say such an edge is *matched*; otherwise say it is *unmatched*. See Figure 4.

If e is matched, an elementary unzip cobordism, Figure 6(Top Center), along e yields a totally oriented foamy cobordism from $\vec{\Gamma}$ to a Klein graph $\vec{\Gamma}'$ with two fewer vertices obtained by unzipping $\vec{\Gamma}$ along e. By induction $\vec{\Gamma}'$ bounds a totally oriented slice foam, and this cobordism extends to a totally oriented slice foam bounded by $\vec{\Gamma}$.

If e is unmatched, an elementary I-H cobordism, shown in Figure 6(Bottom Left), along e yields a totally oriented foamy cobordism from $\vec{\Gamma}$ to a graph $\vec{\Gamma}'$ with the same number of vertices obtained by exchanging e for an edge e'. This edge e' has the same set of edges adjacent to it as e, but partitioned differently at each of its endpoints. This forces e' to have the sign opposite that of e. (One may check the two unmatched cases from Figure 4 directly.) Furthermore, if $\vec{\Gamma}$ has no matched edges, then $\vec{\Gamma}'$ will have a matched edge: If f is an i-edge incident to e in $\vec{\Gamma}$, then at its other end is a k-edge g with sign opposite that of e since f (and every other edge) is unmatched. However in $\vec{\Gamma}'$ the edge g has the same sign as e' so that f is now
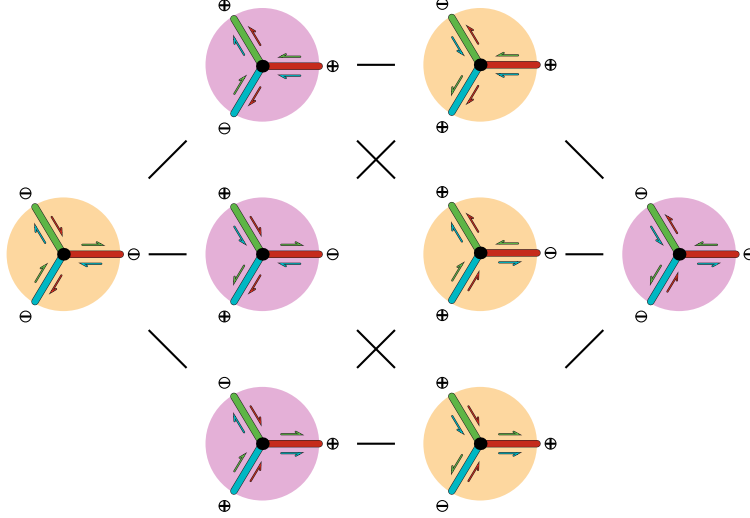
FIGURE 5. The eight total orientations of vertices in a Klein graph are arranged in a cubic graph. Adjacent vertices differ by reversal of one bicolored orientation. Vertices with opposite orientation are at opposite vertices of the cube. The highlighting shows the partition into the two tetrahedral orientation types.

matched. Thus an elementary unzip cobordism may be performed along f in $\overrightarrow{\Gamma}'$ yielding a totally oriented cobordism from $\overrightarrow{\Gamma}$ through $\overrightarrow{\Gamma}'$ to a Klein graph $\overrightarrow{\Gamma}''$ that has two fewer vertices. By induction $\overrightarrow{\Gamma}''$ bounds a totally oriented slice foam, and this cobordism extends it to a totally oriented slice foam bounded by $\overrightarrow{\Gamma}$. □

3.2. **Total orientations, vertices, and seam vertices.** In the following we show that totally oriented Klein graphs are partitioned into totally oriented seamed foamy cobordism classes by a function $\mathrm{sv}(\bullet)$ that is a lift of the function $|\mathrm{sv}(\bullet)|$. Indeed it will follow that for any pair of totally oriented Klein graphs Γ_0, Γ_1, any totally oriented foamy cobordism needs at least $|\mathrm{sv}(\Gamma_0) - \mathrm{sv}(\Gamma_1)|$ seamed vertices (of the appropriate RGB/BGR type), and there is such a foam with exactly that many.

Definition 3.7 (Vertex total orientations, RGB and BGR types)**.** Locally, a vertex of a Klein graph Γ with its three incoming edges admits 8 total orientations. These may be arranged as the vertices of a cube joined by an edge when one bicolored orientation is reversed. See Figure 5. Reversing the total orientation pairs them up as opposite vertices of the cube. One pair has three negative edges and the other three pairs have a single negative edge. A vertex with three negative edges has a cyclic ordering as either RGB or BGR: Say the ordering is RGB if the bicolored arc orientations go $\overrightarrow{rg}$, $\overrightarrow{gb}$, and $\overrightarrow{br}$ where $\overrightarrow{ij}$ indicates that the ij-arc is oriented at the vertex with an incoming i-edge and and outgoing j-edge. In Figure 5, the vertex at the far right has the RGB type while the vertex at the far left has the BGR type.

Definition 3.8 (Signed seam vertex count sv)**.** The tetrahedral graph Γ_{tet} admits 2 total orientations. For each total orientation, the four vertices correspond to one set of the four non-adjacent vertices in the cube graph. See Figure 5. Hence we can also define a totally oriented tetrahedral graph as either RGB or BGR according to the type of its triply negative vertex.

A regular neighborhood of a seam vertex in a foam is a cone F_{vtx} on the tetrahedral graph. Hence a seam vertex of a totally oriented foam is also either RGB or BGR according to the totally oriented tetrahedral graph in the boundary of this cone.

For a totally oriented Klein graph $\overrightarrow{\Gamma}$, let $RGB(\overrightarrow{\Gamma})$ be the number of RGB vertices and let $BGR(\overrightarrow{\Gamma})$ be the total number of BGR vertices. Then define the *signed seam vertex count* to be $\mathrm{sv}(\overrightarrow{\Gamma}) = RGB(\overrightarrow{\Gamma}) - BGR(\overrightarrow{\Gamma})$.

Lemma 3.9. *The quantity* $\mathrm{sv}(\overrightarrow{\Gamma})$ *is unchanged by mirroring, negated by orientation reversal, and additive under connected sum. That is,*

$$\mathrm{sv}(m\overrightarrow{\Gamma}) = \mathrm{sv}(\overrightarrow{\Gamma}) \qquad but \qquad \mathrm{sv}(-\overrightarrow{\Gamma}) = -\mathrm{sv}(\overrightarrow{\Gamma})$$

and

$$\mathrm{sv}(\overrightarrow{\Gamma_1}\#_2\overrightarrow{\Gamma_2}) = \mathrm{sv}(\overrightarrow{\Gamma_1}) + \mathrm{sv}(\overrightarrow{\Gamma_2}) \qquad and \qquad \mathrm{sv}(\overrightarrow{\Gamma_1}\#_3\overrightarrow{\Gamma_2}) = \mathrm{sv}(\overrightarrow{\Gamma_1}) + \mathrm{sv}(\overrightarrow{\Gamma_2}).$$

Proof. The behavior under mirroring and orientation reversal are immediate from the actions of these operations on the signs of the vertices. The additivity follows fairly directly too as an edge connected sum has the same set of oriented vertices as its summands while a vertex connected sum fuses vertices of opposite sign. Indeed, a vertex connected sum may be viewed as a tribanding (Definition 3.10 below) of the split union of the two summands. $\square$

Definition 3.10 (Triband cobordism). Let $\curlyvee$ be the graph in B^3 that is the cone on 3 points in S^2. Given a Klein graph Γ_0, *triband* on Γ is an embedding of $\phi\colon \curlyvee \times I \to S^3$ so that

- the intersection $\phi(\curlyvee \times I) \cap \Gamma_0 = \phi(\curlyvee \times \partial I)$ is a closed neighborhood of a pair of vertices, and
- the three arcs of $\phi(\partial\curlyvee \times I)$ connect edges of the same color.

Let Γ_1 be the Klein graph resulting from surgery along the triband. That is, $\Gamma_1 = (\Gamma_0 - \phi(\curlyvee \times \partial I)) \cup \phi(\partial\curlyvee \times I)$. We say Γ_1 is the result of a *tribanding* of Γ_0 along the two vertices.

In $S^3 \times [0,2]$, the foam $F = \Gamma_0 \times [0,1] \cup \phi(\curlyvee \times I) \times \{1\} \cup \Gamma_1 \times [1,2]$ gives the *triband cobordism* from Γ_0 to Γ_1.

Furthermore, if Γ_0 is endowed with a total orientation and the tribanding is along a pair of oppositely oriented vertices, then the total orientation extends across the triband cobordism giving a total orientation on Γ_1 as well.

Lemma 3.11. *A connected component of a totally oriented Klein graph has either*

- *no vertices,*
- *a pair of oppositely oriented vertices, or*
- *all four types of vertices of tetrahedral orientation.*

Proof. Let v be a vertex of a connected totally oriented Klein graph that has no pair of oppositely oriented vertices. We must then show that the graph has the other three orientation types of vertices of same the tetrahedral orientation of v. Using Figure 5, for any color i, one checks that an i-edge may join v to only two types of oriented vertices, one of them has the opposite orientation while the other belongs to the same tetrahedral orientation (shown as purple and orange in the figure). If there were a pair of edges between two vertices they must have opposite orientation, therefore there is at most one edge between any pair of vertices. Thus v must be adjacent to three different vertices. One further sees that the three vertices adjacent to v must all have different orientation types, but belong to the same tetrahedral orientation. $\square$

Lemma 3.12. *Let Γ be a totally oriented Klein graph. Any totally oriented foam F with Γ as its boundary requires at least $|\mathrm{sv}(\Gamma)|$ seam vertices. Moreover there is a totally oriented foam F with $\partial F = \Gamma$ with exactly $|\mathrm{sv}(\Gamma)|$ seam vertices and these vertices are RGB if $\mathrm{sv}(\Gamma) > 0$ and BGR if $\mathrm{sv}(\Gamma) < 0$.*

Proof. Suppose a totally oriented Klein graph Γ bounds a totally oriented foam F without seam vertices. Using triband cobordisms, one may construct a foamy cobordism F without seam vertices from Γ to a totally oriented Klein graph Γ' where no pairs of vertices have opposite orientation. Observe that $\mathrm{sv}(\Gamma) = \mathrm{sv}(\Gamma')$. By a total orientation reversal if needed, assume $\mathrm{sv}(\Gamma') \geq 0$. By Lemma 3.11, any component of Γ' with a vertex must have a RGB or a BGR vertex an. So Γ' has $\mathrm{sv}(\Gamma')$ RGB vertices (since we have eliminated pairs of vertices of opposite orientation) and in fact $\mathrm{sv}(\Gamma')$ vertices of each other RGB tetrahedral orientation type.

Using a seam vertex we can cap off a vertex and its three neighbors. With $|\mathrm{sv}(\Gamma')|$ applications of this, we arrive at a totally oriented Klein graph without vertices. Such graphs bound totally orientable foams without seam vertices. Put together, Γ bounds a totally oriented foam with exactly $|\mathrm{sv}(\Gamma)|$ seam vertices.

If Γ were to bound a totally oriented foam with fewer seam vertices, then so does Γ'. However then some triple negative RGB vertex of Γ' would have to be the end of a properly embedded seam edge. At the other end would be a vertex with the opposite orientation, contrary to the construction of Γ'. $\square$

3.3. **Cobordisms and connected sums of totally oriented Klein graphs.**

Lemma 3.13. *Suppose $\overrightarrow{F}$ is a totally oriented foamy cobordism in $S^3 \times [0,1]$ from $\overrightarrow{\Gamma_0}$ to $\overrightarrow{\Gamma_1}$ with a spanning seam edge s between vertices v_i of $\overrightarrow{\Gamma}_i$. Then $\overrightarrow{F'} = \overrightarrow{F} \backslash N(s)$ is a totally oriented foam in B^4 with $\partial \overrightarrow{F'} = \overrightarrow{\Gamma_1} \#_3 - m(\overrightarrow{\Gamma}_0)$ where $-m(\overrightarrow{\Gamma}_0)$ is the reversed mirror of $\overrightarrow{\Gamma}_0$ and the connected sum is along the vertices v_i.*

Proof. This follows from basically the same proof as for the case of cobordisms between oriented links. □

Lemma 3.14. *Let $\overrightarrow{\Gamma}$ be a totally oriented Klein graph in S^3. Let v_0 and v_1 be two vertices of the same orientation type. Then there is a seamed foamy cobordism from $\overrightarrow{\Gamma}$ to itself for which the permutation on the vertices induced by the seams is a transposition of v_0 and v_1.*

Proof. Begin with the identity cobordism $\overrightarrow{F} = \overrightarrow{\Gamma} \times I$ in $S^3 \times I$. Let s_i be the seam edge $v_i \times I$, and choose a point p_i in the interior of s_i. Observe that the boundary $S_i = \partial N(p_i)$ of a small regular neighborhood $N(p_i)$ is a 3-sphere that meets $\overrightarrow{F}$ in a trivial θ-curve $\overrightarrow{\theta}_i$ with vertices $S_i \cap s_i$. The total orientation on $\overrightarrow{\theta}_i$ is induced by the foam $\overrightarrow{F} \cap N(p_i)$ in the 4-ball $N(p_i)$ so that the vertex v_i^+ of $S_i \cap s_i$ with higher I–coordinate has the same orientation type as v_i while the other v_i^- has the opposite orientation type.

Now choose a simple arc ϕ in $S^3 \times I$ connecting p_0 to p_1 that is otherwise disjoint from F. Then the ball $N(\phi)$ is the tubing of the two balls $N(p_0)$ and $N(p_1)$ and its boundary $S = \partial N(\phi)$ is the connected sum of S_0 and S_1 that meets F in the split graph $\overrightarrow{\theta}_0 \sqcup \overrightarrow{\theta}_1$. (The total orientation is still induced from $\overrightarrow{F} \cap N(\phi) = \overrightarrow{F} \cap (N(p_0) \cup N(p_1))$.) Since v_0 and v_1 have the same orientation type, $\overrightarrow{\theta}_0$ and $\overrightarrow{\theta}_1$ are isomorphic as totally oriented Klein graphs. Furthermore, because $\overrightarrow{\theta}_0$ is a trivial θ-curve, it is isotopic to its own reverse. Because they are embedded as a split graph, there is a totally oriented Klein foam $\overrightarrow{F_\theta}$ homeomorphic to $\overrightarrow{\theta}_0 \times I$ and embedded in S with $\partial \overrightarrow{F_\theta} = \overrightarrow{\theta}_0 \sqcup \overrightarrow{\theta}_1$ so that the seams of $\overrightarrow{F_\theta}$ connect v_0^+ to v_1^- and v_0^- to v_1^+. Replacing $\overrightarrow{F} \cap N(\phi)$ with $\overrightarrow{F_\theta}$ produces a new totally oriented Klein foam $\overrightarrow{F'}$ in which s_0 and s_1 have been replaced by seams that connect one end of $v_0 \times \partial I$ to the other end of $v_1 \times \partial I$ and vice-versa, leaving all other seams as they were and producing the desired permutation. □

Lemma 3.15. *Suppose there is a seamed foamy cobordism from $\overrightarrow{\Gamma}_0$ to $\overrightarrow{\Gamma}_1$. Then for any pair of vertices v_i of $\overrightarrow{\Gamma}_i$ with the same orientation type, there exists a seamed foamy cobordism from $\overrightarrow{\Gamma}_0$ to $\overrightarrow{\Gamma}_1$ with seam edge connecting v_0 to v_1.*

Proof. This follows after applying Lemma 3.14 to construct a seamed foamy cobordism from $\overrightarrow{\Gamma}_1$ to itself yielding a permutation of the vertices v_0 and v_1. □

Lemma 3.16. *Suppose $\overrightarrow{\Gamma}_0$ and $\overrightarrow{\Gamma}_1$ are totally oriented Klein graphs. For any given pair of vertices v_i of $\overrightarrow{\Gamma}_i$ with the same orientation type, form the connected sum $\overrightarrow{\Gamma_1} \#_3 - m\overrightarrow{\Gamma}_0$ along v_0 and v_1. If there is a seamed foamy cobordism between $\overrightarrow{\Gamma}_0$ and $\overrightarrow{\Gamma}_1$, then there is a slice foam for $\overrightarrow{\Gamma_1} \#_3 - m\overrightarrow{\Gamma}_0$.*

Proof. This proof essentially follows the analogous construction of a slice disk for $K_1 \# - mK_0$ from a concordance between K_0 and K_1. First, by Lemma 3.15, there exists a seamed foamy cobordism $\overrightarrow{F}$ from $\overrightarrow{\Gamma}_0$ to $\overrightarrow{\Gamma}_1$ with a seam edge e connecting v_0 to v_1. Then $\overrightarrow{F} \backslash \mathcal{N}(e)$ is then a slice foam for $\overrightarrow{\Gamma_1} \#_3 - m\overrightarrow{\Gamma}_0$. □

4. Bounds

In this section, we prove Theorem 4.1, bounding the orbifold Euler characteristic of the graph $\Gamma_1 \#_3 - m\Gamma_2$ below by the signature difference, and above by the Klein Gordian distance between Γ_1 and Γ_2.

Theorem 4.1. *Suppose that $\overrightarrow{\Gamma_1}$ and $\overrightarrow{\Gamma_2}$ are totally oriented Klein graphs that are related by a sequence of crossing changes. Set $V := |V(\Gamma_1)| = |V(\Gamma_2)|$ and $\mu := \mu(\Gamma_1) = \mu(\Gamma_2)$. Then:*

$$|\sigma(\overrightarrow{\Gamma_1}) - \sigma(\overrightarrow{\Gamma_2})| - \beta(\overrightarrow{\Gamma_1}) - \beta(\overrightarrow{\Gamma_2}) + 4\mu - 12 \overset{(a)}{\leq} 5 - 4\chi_4^{orb}(\overrightarrow{\Gamma_1} \#_3 - m\overrightarrow{\Gamma_2}) - 2V$$

$$\overset{(b)}{\leq} -4\chi_4^{orb}(\overrightarrow{\Gamma_1}, \overrightarrow{\Gamma_2}; s) - 2V$$

$$\overset{(c)}{\leq} 4d_\curlywedge(\overrightarrow{\Gamma_1}, \overrightarrow{\Gamma_2})$$

where $\overrightarrow{\Gamma_1} \#_3 - m\overrightarrow{\Gamma_2}$ is any vertex connected sum of $\overrightarrow{\Gamma_1}$ and $-m\overrightarrow{\Gamma_2}$ compatible with the total orientations.

Proof. In subsection 4.1, Corollary 4.6 proves inequality (a) since both graphs will have the same signed vertex count sv and Lemma 4.7 establishes inequality (b). In subsection 4.2 we prove inequality (c). The proof then follows. □

Proof of Corollary 1.1. Let $\overrightarrow{\Gamma_1}$ be θ with any Klein coloring and total orientation. Let $\overrightarrow{\Gamma_2}$ be a totally oriented and Klein colored trivial theta with the same vertex types as $\overrightarrow{\Gamma_1}$. Then $\overrightarrow{\Gamma_1} \# _3 - m\overrightarrow{\Gamma_2} = \overrightarrow{\Gamma_1}$.

We now apply Theorem 4.1. Observe that the signature of the trivial theta vanishes, and for any totally oriented, Klein colored theta curve we have $\beta = 0$, $4\mu - 12 = 0$, and $5 - 2V = 1$. Finally, $d_\curlywedge(\overrightarrow{\Gamma_1}, \overrightarrow{\Gamma_2})$ is $u_\curlywedge(\theta)$ since unknotting a theta curve is insensitive to total orientation and any change of Klein coloring preserves the same and mixed crossing types. So, inequalities (a) and (c) give the desired result. □

Remark 4.2. The inequality $\frac{1}{4}|\sigma(\theta)| \leq u_\curlywedge(\theta)$ for θ-curves from Corollary 1.1 (without the slice orbifold Euler characteristic), can be obtained by careful application of Theorem 2.19, and $\frac{1}{2}|\sigma(K)| \leq u(K)$ for a knot K.

We begin by exploring inequalities relating signature to orbifold Euler characteristic and then establish the inequalities relating orbifold Euler characteristic to sequences of crossing changes.

4.1. **Bounding signature.** To establish the upper bound on the signature of a Klein graph, we use the corresponding result for knots. To establish the upper bound on the difference of signatures of two Klein graphs, we use properties of the vertex sum.

Setting $z = -1$ in [Pow17, Theorem 1.4], Powell shows that $|\sigma(\overrightarrow{L})| + \mu(L) - 1 - \beta(\overrightarrow{L}) \leq 2g_4(\overrightarrow{L})$ so that

$$|\sigma(\overrightarrow{L})| \leq 2g_4(\overrightarrow{L}) - \mu(L) + 1 + \beta(\overrightarrow{L})$$
$$= (2g_4(\overrightarrow{L}) + \mu(L) - 2) - 2\mu(L) + 3 + \beta(\overrightarrow{L})$$
$$= 1 - \chi_4(\overrightarrow{L}) - 2(\mu(L) - 1) + \beta(\overrightarrow{L}).$$

Theorem 4.3. *For a totally oriented Klein graph $\overrightarrow{\Gamma} \subset S^3$, we have*

$$|\sigma(\overrightarrow{\Gamma})| \leq 3 - |V(\Gamma)| + 2|\mathrm{sv}(\overrightarrow{\Gamma})| - 4\chi_4^{orb}(\overrightarrow{\Gamma}) - 2(\mu(\Gamma) - 3) + \beta(\overrightarrow{\Gamma}).$$

Proof. By the definition of χ_4^{orb}, there exists a totally orientable foamy cobordism $\overrightarrow{F}$ for $\overrightarrow{\Gamma}$ such that $\overrightarrow{F}$ has no bicolored spheres and $\chi^{orb}(\overrightarrow{F}) = \chi_4^{orb}(\overrightarrow{\Gamma})$.

By Theorem 2.19:

$$|\sigma(\overrightarrow{\Gamma})| = |\sigma(\overrightarrow{\Gamma}_{rb}) + \sigma(\overrightarrow{\Gamma}_{bg}) + \sigma(\overrightarrow{\Gamma}_{rg})|$$
$$\leq |\sigma(\overrightarrow{\Gamma}_{rb})| + |\sigma(\overrightarrow{\Gamma}_{bg})| + |\sigma(\overrightarrow{\Gamma}_{rg})|$$

Since each $\overrightarrow{\Gamma}_{ij}$ is an oriented link, and $|\sigma(\overrightarrow{L})| \leq 1 - \chi_4(\overrightarrow{L}) - 2(\mu(L) - 1) + \beta(\overrightarrow{L})$ for any oriented link $\overrightarrow{L}$ by [Pow17, Theorem 1.4], then

$$\leq 3 - (\chi_4(\overrightarrow{\Gamma}_{rb}) + \chi_4(\overrightarrow{\Gamma}_{bg}) + \chi_4(\overrightarrow{\Gamma}_{rg})) - 2(\mu(\Gamma) - 3) + \beta(\overrightarrow{\Gamma})$$

Since F is totally oriented with no bicolored spheres, each surface $\overrightarrow{F}_{ij}$ is oriented and has no sphere components. Then, since $-\chi_4(\overrightarrow{\Gamma}_{ij}) \leq -\chi(\overrightarrow{F}_{ij})$ by definition,

$$-\left(\chi_4(\overrightarrow{\Gamma}_{rb}) + \chi_4(\overrightarrow{\Gamma}_{bg}) + \chi_4(\overrightarrow{\Gamma}_{rg})\right) \leq -(\chi(F_{rb}) + \chi(F_{bg}) + \chi(F_{rg}))$$

Since each seam edge lies on all three bicolored surfaces and each monochromatic face lies on two bicolored surfaces:

$$\chi(F_{rb}) + \chi(F_{bg}) + \chi(F_{rg}) = (2(\chi(F_r) + \chi(F_b) + \chi(F_g)) - 3\chi(s(F)))$$
$$= 2(\chi(F) + 2\chi(s(F))) - 3\chi(s(F))$$
$$= 2\chi(F) + \chi(s(F))$$

By Definition 2.13, $4\chi^{orb}(F) = 2\chi(F) - \chi(s(F))$. So:

$$= 4\chi^{orb}(F) + 2\chi(s(F))$$

15

As $s(F)$ is a 1-complex with $|V| := |V(\Gamma)|$ boundary vertices and $|sv| := |sv(\vec{\Gamma})|$ seam vertices, it has $(|V| + 4|sv|)/2$ edges. So $2\chi(s(F)) = 2(|V| + |sv|) - (|V| + 4|sv|) = |V| - 2|sv|$. So:

$$= 4\chi^{orb}(F) + |V| - 2|sv|$$
$$= 4\chi_4^{orb}(\vec{\Gamma}) + |V| - 2|sv|$$

Therefore we have

$$|\sigma(\vec{\Gamma})| \leq 3 - (4\chi_4^{orb}(\vec{\Gamma}) + |V| - 2|sv|) - 2(\mu(\Gamma) - 3) + \beta(\vec{\Gamma})$$
$$\leq 3 - |V(\Gamma)| + 2|sv| - 4\chi_4^{orb}(\Gamma) - 2(\mu(\Gamma) - 3) + \beta(\vec{\Gamma})$$

as claimed. $\qquad\square$

Lemma 4.4. *Suppose that $\vec{\Gamma}$, $\vec{\Gamma_1}$, and $\vec{\Gamma_2}$ are totally oriented Klein graphs. Then:*

$$\sigma(-\vec{\Gamma}) = \sigma(\vec{\Gamma}) \qquad and \qquad \sigma(m\vec{\Gamma}) = -\sigma(\vec{\Gamma})$$

and

$$\sigma(\vec{\Gamma_1}\#_3\vec{\Gamma_2}) = \sigma(\vec{\Gamma_1}) + \sigma(\vec{\Gamma_2})$$

for any vertex sum compatible with the total orientations.

Proof. As oriented links, $(\Gamma_1\#_3\Gamma_2)_{ij} = (\Gamma_1)_{ij}\#(\Gamma_2)_{ij}$ for $i, j \in \{r, g, b\}$. Then this lemma follows from Corollary 2.27 and the fact that signature is unchanged under orientation reversal, negated under mirroring, and additive for oriented links (e.g. [BZH14, Proposition 13.12] and [CF08, Proposition 2.13]). $\qquad\square$

This immediately gives:

Corollary 4.5. *Suppose that $\vec{\Gamma_1}$ and $\vec{\Gamma_2}$ are totally oriented Klein graphs. Then:*

$$\sigma(\vec{\Gamma_1}) - \sigma(\vec{\Gamma_2}) = \sigma(\vec{\Gamma_1}\#_3 - m\vec{\Gamma_2})$$

where $\vec{\Gamma_1}\#_3 - m\vec{\Gamma_2}$ is any vertex sum of $\vec{\Gamma_1}$ and $-m\vec{\Gamma_2}$ compatible with the total orientations. $\qquad\square$

Corollary 4.6. *Suppose that $\vec{\Gamma_1}$ and $\vec{\Gamma_2}$ are totally oriented Klein graphs, each with at least one vertex. Let $\mu_i = \mu(\vec{\Gamma_i})$ and $V_i = |V(\vec{\Gamma_i})|$. Then:*

$$|\sigma(\vec{\Gamma_1}) - \sigma(\vec{\Gamma_2})| \leq 3 - (V_1 + V_2 - 2) + 2|sv(\vec{\Gamma_1}) - sv(\vec{\Gamma_2})| - 4\chi_4^{orb}(\vec{\Gamma_1}\#_3 - m\vec{\Gamma_2}) - 2(\mu_1 + \mu_2 - 6) + \beta(\vec{\Gamma_1}) + \beta(\vec{\Gamma_2})$$

where $\Gamma_1\#_3 - m\Gamma_2$ is any vertex sum of $\vec{\Gamma_1}$ and $-m\vec{\Gamma_2}$ compatible with the total orientations.

Proof. This follows from Theorem 4.3 and Corollary 4.5 along with Lemmas 2.24, 2.25, and 3.9. $\qquad\square$

Lemma 4.7. *Suppose that $\vec{\Gamma_1}$ and $\vec{\Gamma_2}$ are totally oriented Klein graphs, each with at least one vertex, that cobound a totally orientable seamed foamy cobordism. Then,*

$$\chi_4^{orb}(\vec{\Gamma}_1, \vec{\Gamma}_2; s) \leq \chi_4^{orb}(\vec{\Gamma_1}\#_3 - m\vec{\Gamma_2}) - \frac{5}{4}$$

where $\vec{\Gamma_1}\#_3 - m\vec{\Gamma_2}$ is any vertex sum of $\vec{\Gamma_1}$ and $-m\vec{\Gamma_2}$ compatible with the total orientations.

Proof. Let $\curlyvee$ be the graph in B^3 that is the cone on 3 points in S^2. Let F be a seamed foamy cobordism from $\vec{\Gamma}_1$ to $\vec{\Gamma}_2$ in $S^3 \times [0, 1]$ realizing $\chi_4^{orb}(\vec{\Gamma}_1, \vec{\Gamma}_2; s)$ and having no bicolored spheres.

By Lemma 3.16, from any pair of vertices $v_1 \in \vec{\Gamma}_1$ and $v_2 \in \vec{\Gamma}_2$ with the same orientation, we may construct a slice foam for $\vec{\Gamma_1}\#_3 - m\vec{\Gamma_2}$ by summing along a seam connecting v_1 and v_2.

Using Definition 2.13, we may count

$$\chi^{orb}(F) = \chi^{orb}(F') + \frac{1}{4} - \frac{3}{2} = \chi^{orb}(F') - \frac{5}{4}.$$

Since F' may not realize $\chi_4^{orb}(\vec{\Gamma_1}\#_3 - m\vec{\Gamma_2})$, we then have the stated inequality. $\qquad\square$

Remark 4.8. A priori, vertex sums along different pairs of vertices may produce different foamy cobordisms. The slice orbifold Euler characteristic that we calculate thus depends on the choice of vertices, but for any choice the slice orbifold characteristic is bounded below by the signature difference.
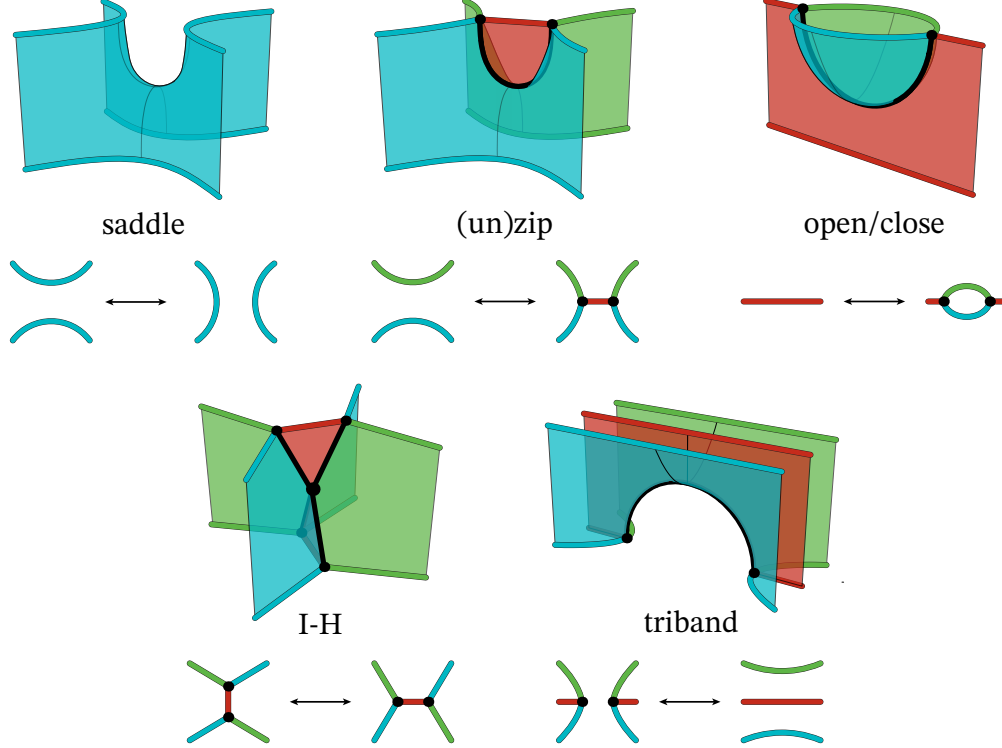
FIGURE 6. (Top Left) The elementary saddle cobordism. (Top Center) The elementary (un)zip cobordism. (Top Right) The elementary open/close cobordism. (Bottom Left) The elementary I-H cobordism. (Bottom Right) The triband cobordism.

4.2. **Building foams from crossing changes.** In this section, we build foams with a view exploring the inequality (c) in Theorem 4.1. The two basic elementary cobordisms we employ are shown in Figure 6(Top Left) and (Top Center): the elementary saddle cobordism and the elementary (un)zip cobordism.

Theorem 4.9. *Suppose there is a sequence of s same-colored crossing changes and m mixed-colored crossing changes between totally oriented Klein graphs $\overrightarrow{\Gamma_1}$ and $\overrightarrow{\Gamma_2}$. Then there is a totally orientable seamed foamy cobordism $\overrightarrow{F}$ between $\overrightarrow{\Gamma_1}$ and $\overrightarrow{\Gamma_2}$ without bicolored spheres with*

$$\chi^{orb}(\overrightarrow{F}) = -|V(\overrightarrow{\Gamma_1})|/2 - (s + m/2).$$

Remark 4.10. Crossing changes do not alter total orientations on the edges involved. Indeed, any choice of total orientation on Γ_1 confers a total orientation on Γ_2 and extends to a total orientation on the seamed foamy cobordism F constructed in the proof of Theorem 4.9.

Proof of Theorem 4.9. In the appendix of [GR18], a foam is constructed from a sequence of crossing changes on a Klein graph. However, the constructed foam is not necessarily totally orientable. Here we review such a construction and how to alter it in order to produce a totally orientable foamy cobordism. The key step is to replace any problematic same-colored crossing change by a pair of mixed-colored crossing changes. Thereafter, we determine the Euler characteristic of the constructed foam.

Mirroring as needed, Figure 7 shows, via a movie, how single same-colored or mixed-colored crossing changes of oriented arcs of a Klein graph can be realized as an "elementary" foamy cobordism of type F_s or F_m from Γ to Γ_s or Γ_m, respectively. Observe that F_s is assembled using two elementary saddle cobordisms while F_m is assembled using a pair of elementary zip/unzip cobordisms. See Figure 6. Further note that both F_s and F_m are seamed foamy cobordisms.

Stacking these cobordisms together according to a crossing change sequence from a totally oriented Klein graph $\overrightarrow{\Gamma_1}$ to a totally oriented Klein graph $\overrightarrow{\Gamma_2}$ then assembles a seamed foamy cobordism F between the two
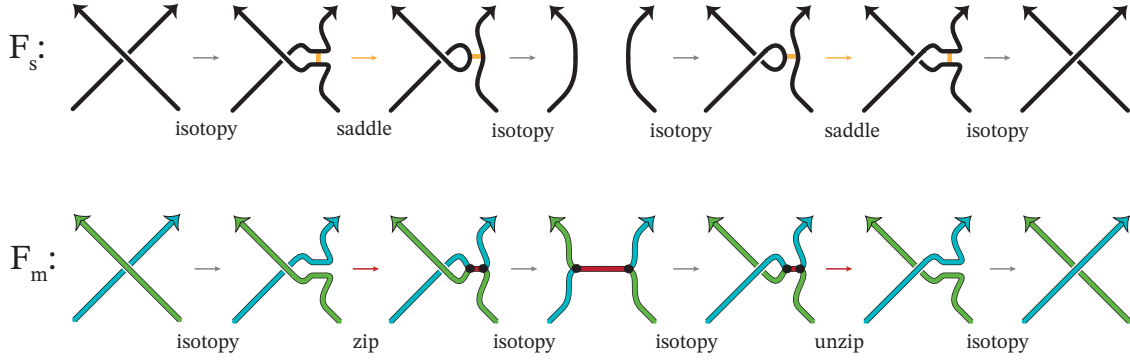
FIGURE 7. Top: Movie of an oriented same crossing change cobordism F_s realized by two elementary saddle cobordisms. Bottom: Movie of an oriented mixed crossing change cobordism F_m realized by a pair of elementary zip/unzip cobordisms.
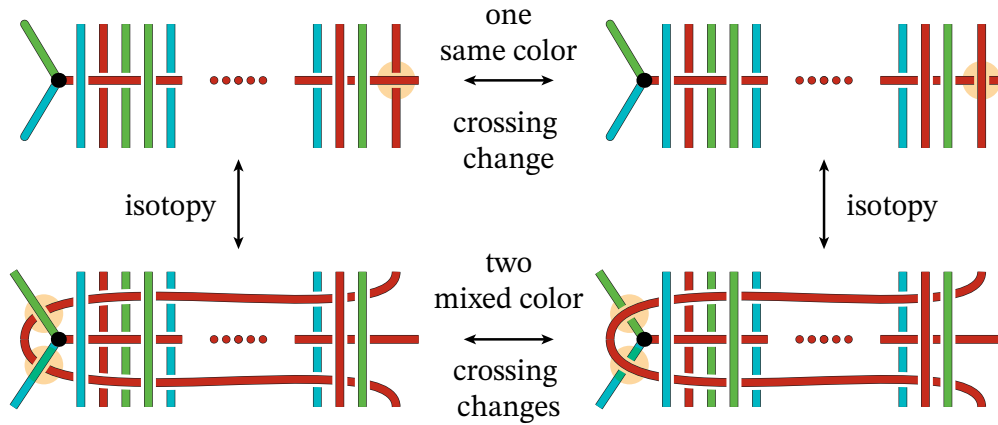

FIGURE 8. A same crossing change may be realized as two mixed crossing changes.

graphs. However, one must be more careful to ensure the foamy cobordism F admits a total orientation that respects the total orientations of the two graphs.

Observe that crossing changes do not alter the double orientations on edges of a totally oriented Klein graph. Hence, through the sequence of crossing changes, the total orientation on Γ_1 induces a total orientation on each of the intermediate graphs and is carried to the total orientation on Γ_2. To make the elementary crossing change foams piece together into a totally oriented foam, the elementary crossing change cobordisms need to respect these total orientations.

Figure 7 (Top) shows how a saddle-saddle sequence produces a foamy cobordism associated to a same-color crossing change of oriented arcs. This same-color crossing change cobordism F_s between Klein graphs Γ and Γ_s is built from two saddle moves that respect a choice of orientations on the arcs involved. Together these saddles form a 2-dimensional 1-handle that is attached in a manner that respects the orientation chosen on these arcs of Γ. (The slice genus bound on unknotting number for knots follows from stacking such cobordisms.)

Note that a same-colored crossing change involves either one or two edges. When the double orientations of the edges involved in a same-colored crossing change have the same sign (are all parallel or all antiparallel), one may observe that F_s is totally orientable in a manner that respects the total orientations of Γ and the resulting Γ_s. However, suppose a same-colored crossing change involved two distinct i-colored edges where the edges have opposite sign. Then one of the surfaces $(F_s)_{ij}$ and $(F_s)_{ik}$ is orientable while the other is non-orientable. Hence, the cobordism F_s associated to a same-color crossing change is totally orientable

18

(with a total orientation respecting the total orientations of Γ and Γ_s) exactly when the edges involved have the same sign of double orientation. Fortunately, as shown in Figure 8, a problematic same-color crossing change (where the edges have different signs) can be realized as a pair of mixed-color crossing changes.

Figure 7 (Bottom) shows how a zip-unzip sequence produces a seamed foamy cobordism associated to a mixed-color crossing change of oriented arcs. This mixed-color crossing change cobordism $\overrightarrow{F_m}$ between totally oriented Klein graphs $\overrightarrow{\Gamma}$ and $\overrightarrow{\Gamma_m}$ is homeomorphic to the totally oriented trivial cobordism $\overrightarrow{\Gamma} \times I$ with a 1-handle joining the facet surfaces of the two colors of the crossing change along with a disk of the third color that is the co-core of the handle. (This move adds a closed component to the seams.) Suppose the mixed-color crossing changes involves an i-colored and a j-colored edge. When forming $\overrightarrow{F_m}$, the 1-handle is attached to $\overrightarrow{\Gamma} \times I$ so that $(\overrightarrow{F_m})_{ij}$ is oriented, extending the orientation of $(\overrightarrow{\Gamma} \times I)_{ij}$. Then, since $(\overrightarrow{F_m})_{ik}$ and $(\overrightarrow{F_m})_{jk}$ are identified with $(\overrightarrow{\Gamma} \times \overrightarrow{I})_{ik}$ and $(\overrightarrow{\Gamma} \times \overrightarrow{I})_{jk}$ by having an interior disk of $(\overrightarrow{\Gamma} \times I)_i$ and $(\overrightarrow{\Gamma} \times I)_j$ changed to a k-colored disk, the surfaces are oriented. Indeed, this shows that the bicolored surfaces of $\overrightarrow{F_m}$ are oriented in a manner that respect the total orientations of Γ and Γ_m. Hence $\overrightarrow{F_m}$ is a totally oriented cobordism between $\overrightarrow{\Gamma}$ and $\overrightarrow{\Gamma_m}$.

Now, given a sequence of crossing changes from $\overrightarrow{\Gamma_1}$ to $\overrightarrow{\Gamma_2}$, exchange any same-colored crossing change between a parallel edge and an antiparallel edge for a pair of mixed-colored crossings changes so that the crossing changes are realized by totally oriented elementary foamy cobordisms. Then stack all these totally oriented elementary foamy cobordisms together to assemble a totally oriented foamy cobordism $\overrightarrow{F}$ from $\overrightarrow{\Gamma_1}$ to $\overrightarrow{\Gamma_2}$.

Furthermore, one readily observes that each component of each bicolored surface of a mixed-color crossing change cobordism $\overrightarrow{F_m}$ between $\overrightarrow{\Gamma}$ and $\overrightarrow{\Gamma_m}$ has boundary meeting each $\overrightarrow{\Gamma}$ and $\overrightarrow{\Gamma_m}$. Indeed, the only component of $(\Gamma_m)_{ij}$ which is not identified with a component of $(\overrightarrow{\Gamma} \times I)_{ij}$ is the one obtained by a 1-handle attachment. Similarly, each component of each bicolored surface of a totally oriented same-colored crossing change cobordism $\overrightarrow{F_s}$ meets both $\overrightarrow{\Gamma}$ and $\overrightarrow{\Gamma_s}$. Consequently, a totally oriented foamy cobordism from $\overrightarrow{\Gamma_1}$ to $\overrightarrow{\Gamma_2}$ obtained by stacking totally oriented elementary crossing change cobordisms has no bicolored spheres.

Now we calculate the Euler characteristics of these elementary cobordisms. (Total orientations play no role here.) According to the above descriptions, the orbifold Euler characteristics of these elementary crossing change cobordisms are calculated to be

$$\chi^{orb}(F_s) = \chi^{orb}(\Gamma \times I) - \frac{1}{2} \cdot 2 = \chi^{orb}(\Gamma \times I) - 1$$

and

$$\chi^{orb}(F_m) = \chi^{orb}(\Gamma \times I) - \frac{1}{2} \cdot 1 = \chi^{orb}(\Gamma \times I) - \frac{1}{2}.$$

So now given a sequence of s same-colored crossing changes and m mixed-colored crossing changes between Klein graphs Γ_1 and Γ_2 where the edges of Γ_1 have been assigned some orientations, let F be the foamy cobordism from Γ_1 to Γ_2 obtained by stacking the associated elementary crossing change cobordisms. Hence

$$\chi^{orb}(F) = \chi^{orb}(\Gamma_1 \times I) - s - m/2.$$

Finally, the seam edges join the vertices of the components of $\Gamma_1 \times \partial I$, so $\chi^{orb}(\Gamma_1 \times I) = \chi(\Gamma_1)$. Hence, $\chi^{orb}(F) = \chi(\Gamma_1) - s - m/2 = -\frac{1}{2}|V(\Gamma_1)| - s - m/2$, as claimed.

Observe that the contribution to $\chi^{orb}(F)$ of an elementary same-color crossing change cobordism is twice that of a mixed-color crossing change cobordism. Since each same-colored crossing change may be realized by two mixed-color crossing changes, we may instead build a totally oriented foamy cobordism $\overrightarrow{F'}$ from $\overrightarrow{\Gamma_1}$ to $\overrightarrow{\Gamma_2}$ by stacking $m + 2s$ elementary mixed-color crossing change cobordisms so that $\chi^{orb}(\overrightarrow{F'}) = \chi^{orb}(\overrightarrow{F})$. Furthermore, as discussed above, such a totally oriented foamy cobordism $\overrightarrow{F'}$ will have no bicolored spheres. $\quad\square$

Corollary 4.11. *Suppose the Klein Gordian distance between two Klein graphs Γ_1 and Γ_2 is $d_{\curlywedge}(\Gamma_1, \Gamma_2)$. Then*

$$-\chi_4^{orb}(\Gamma_1, \Gamma_2; s) \leq d_{\curlywedge}(\Gamma_1, \Gamma_2) + \frac{1}{2}|V(\Gamma_1)|.$$

Proof. Say $d_\curlywedge(\Gamma_1, \Gamma_2)$ is realized by a sequence of s same-colored crossing changes and m mixed-colored crossing changes. Then by Theorem 4.9 there is a seamed foamy cobordism F such that $\chi^{orb}(F) = -\frac{1}{2}V(\Gamma_1) - d_\curlywedge(\Gamma_1, \Gamma_2)$. Since $\chi^{orb}(F) \leq \chi_4^{orb}(\Gamma_1, \Gamma_2; s)$ by Definition 2.13, we have $-\frac{1}{2}V(\Gamma_1) - d_\curlywedge(\Gamma_1, \Gamma_2) \leq \chi_4^{orb}(\Gamma_1, \Gamma_2; s)$. The inequality follows. $\square$

5. Examples

We present several examples of how our bounds shed light on the unknotting number of θ-curves; our bounds improve previously known bounds. We conclude by presenting examples of θ-curves showing that the stronger form of Gille-Robert's signature does not produce bounds on unknotting number.

5.1. **Improved bounds.** Let the standard unknotting number $u(\theta)$ be the minimum number of crossing changes needed to unknot θ.

This is known to be bounded below by the maximal constituent unknotting number of the graph, that is $mcu(g) = max\{u(s)|$ where s is a constituent of $g\}$ [BO18]. (For any Klein coloring of a θ-curve, a constituent knot is a bicolored knot.) The following is a family of θ-curves where our new bound $\frac{1}{4}|\sigma(\theta)| \leq u_\curlywedge(\theta)$ significantly improves on the past result.
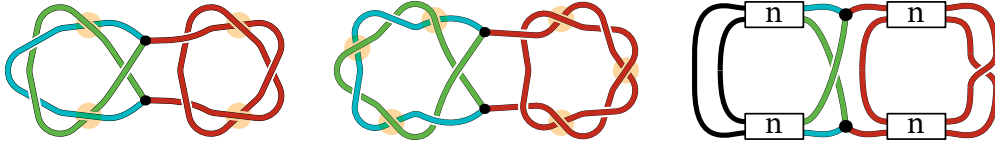

FIGURE 9. (Left) A θ-curve θ_2 for which every constituent knot is the $(2,5)$-torus knot 5_1. Highlighted are a set of 4 crossing changes that unknot θ_2. (Center) A θ-curve θ_3 for which every constituent knot is the $(2,7)$-torus knot 7_1. Highlighted are a set of 6 crossing changes that unknot θ_3. (Right) A generalization to a family of θ-curves θ_n for which every constituent knot is the $(2, 2n+1)$-torus knot.

In the theta curve θ_2 of Figure 9(Left), all of the constituent knots are 5_1, so $mcu(\theta_2) = u(5_1) = 2$. (Note $|\sigma(5_1)| = 4$.) Thus the earlier bounds gives $2 \leq u(\theta_2)$. Next $|\sigma(\theta_2)| = 3|\sigma(5_1)| = 3 \cdot 4 = 12$. So our new bound gives $\frac{1}{4}|\sigma(\theta_2)| = 3 \leq u_\curlywedge(\theta_2)$. In the figure there is a set of 2 same crossings and 2 mixed crossing that will unknot θ_2, giving $s + \frac{m}{2} = 3$. Hence $u_\curlywedge(\theta_2) = 3$ and our lower bound on $u_\curlywedge$ is sharp in this case. Furthermore, $u_\curlywedge(\theta_2) = 3$ implies that $u(\theta_2) \geq 3$, improving the lower bound on unknotting number from 2 to 3.

Similarly, in the theta curve θ_3 of Figure 9(Center), all of the constituent knots are 7_1, so $mcu(\theta_3) = u(7_1) = 3$. (Note $|\sigma(7_1)| = 6$.) Thus the earlier bounds gives $3 \leq u(\theta_3)$. Next $|\sigma(\theta_3)| = 3|\sigma(7_1)| = 3 \cdot 6 = 18$. So our new bound gives $\frac{1}{4}|\sigma(\theta_3)| = 4.5 \leq u_\curlywedge(\theta_3)$. In the figure there is a set of 3 same crossings and 3 mixed crossing that will unknot θ_3, giving $s + \frac{m}{2} = 4.5$. Hence $u_\curlywedge(\theta_3) = 4.5$ and our lower bound on $u_\curlywedge$ is sharp in this case. Furthermore, $u_\curlywedge(\theta_3) = 4.5$ implies that $u(\theta_3) \geq 5$, improving the lower bound on unknotting number from 3 to 5.

Generalizing these examples further gives a family of θ-curves θ_n for $n \in \mathbb{N}$ where all of the constituent knots are torus knots $T(2, 2n+1)$, shown in Figure 9(Right). Here $mcu(\theta_n) = u(T(2, 2n+1)) = n$. Thus the earlier bound gives $n \leq u(\theta_n)$ and our new bound gives $\frac{1}{4}|\sigma(\theta_n)| = \frac{3}{2}n \leq u_\curlywedge(\theta_n)$. Similarly these examples have a set of n same crossings and n mixed crossing that will unknot θ_n, giving $s + \frac{m}{2} = \frac{3}{2}n$. Hence $u_\curlywedge(\theta_n) = \frac{3}{2}n$ while $u(\theta_n) \leq 2n$.

Question 5.1. *While $u(\theta_n) \leq 2n$, our count $u_\curlywedge(\theta_n) = \frac{3}{2}n$ implies that $u(\theta_n) \geq \lceil \frac{3}{2}n \rceil$. Is $u(\theta_n) < 2n$ for some $n \geq 2$?*

Remark 5.2. Observe that $u(\theta_2) \in \{3, 4\}$. If $u(\theta_2) = 3$, then θ_2 would be unknotted by a set of 3 same crossing changes.

Similarly, $u(\theta_3) \in \{5, 6\}$. If $u(\theta_3) = 5$, then θ_3 would be unknotted by a set of 4 same crossing changes and 1 mixed crossing change.

Remark 5.3. Each θ_n is the edge connected sum of a theta curve θ'_n and the knot $T(2, 2n+1)$. One observes that $mcu(\theta'_n) = u(T(2, 2n+1)) = n$ so both of these have unknotting number n. Hence if $u(\theta_n) < 2n$ for some $n \geq 2$, answering Question 5.1 in the affirmative, then the unknotting number for theta curves would be not additive under edge connected sum.

5.2. **Strong signature.** In their article, Gille-Robert define two signature invariants, σ and $\tilde{\sigma}$ for 3-Hamiltonian Klein graphs; we term them the *weak signature* and the *strong signature*. The signature used through this article (which we extend to all Klein graphs) is the weak signature $\sigma(\Gamma)$, defined as

$$\sigma(\Gamma) = \sigma(W_F) + \frac{1}{2}e(F),$$

and is equivalent to the sum of the signatures of the constituent knots $\sigma(\Gamma) = \sigma(K_{rb}) + \sigma(K_{bg}) + \sigma(K_{rg})$ [GR18, Proposition 3.14]. The term $\sigma(W_F)$ is the signature of the 4-manifold Klein cover of the 4-ball branched over a Klein foam F for Γ, and $e(F)$ is the *weak* Euler number of the foam F, given by the sum of relative normal Euler numbers of the facets, $e(F_{rb}) + e(F_{bg}) + e(F_{rg})$. The *strong* signature is defined as

$$\tilde{\sigma}(\Gamma) = \sigma(W_F) + \frac{1}{2}\tilde{e}(F),$$

where $\tilde{e}(F)$ is the *strong* Euler number [GR18, Definition 3.10]. The strong Euler number is given by the sum of relative normal Euler numbers of the surfaces $\tilde{F}^r_r + \tilde{F}^b_b + \tilde{F}^g_g$. Each surface $\tilde{F}^i_i$, $i = r, g, b$ is the image of the fixed point surface of the diffeomorphism induced by the action on W_F of the element i in the Klein group. It is shown in [GR18, Proposition 3.14] that $\tilde{\sigma}(\Gamma)$ can be calculated from a sum $\tilde{\Gamma} = \xi(\tilde{\Gamma}^r_r) + \xi(\tilde{\Gamma}^b_b) + \xi(\tilde{\Gamma}^g_g)$ of signatures of the links $\tilde{\Gamma}^i_i$ in the rational homology spheres bounding the 4-manifolds W/i (which we can also think of as double covers of the 4-ball branched over the surfaces F_{jk}). See [GR18, Defn 3.10], [GR18, Thm 3.11], [GR18, Definition 3.12].

It is natural to ask whether the bounds of Theorem 4.1 extend to strong signature. In the following example, we use the Kinoshita-Wolcott family of theta-curves [Kin58, Wol87] to show that, unlike the weak signature, strong signature does not yield a successful bound.
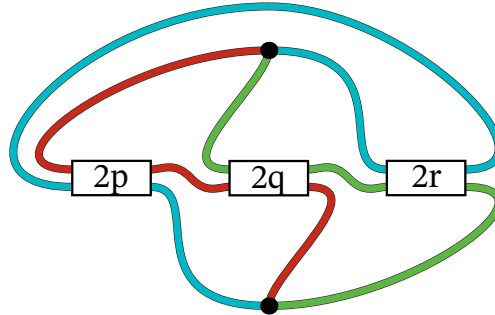

FIGURE 10. The Kinoshita-Wolcott θ-curve $\theta(p, q, r)$. [Kin58, Wol87]

Let $\theta(p, q, r)$ be the Kinoshita-Wolcott θ-curve with p, q and r full twists shown in Figure 10. Notice that all constituent knots are unknotted, thus all edges lift to knots in the three-sphere, which we may use for the calculation of strong signature. Then:

$$\tilde{\sigma}(\theta(p, q, r)) = \begin{cases} -8(p+q+r) & \text{if } p, q, \text{ and } r \text{ are odd} \\ -8p & \text{if } p \text{ is odd and } q \text{ and } r \text{ are even} \\ -8q & \text{if } q \text{ is odd and } p \text{ and } r \text{ are even} \\ -8r & \text{if } r \text{ is odd and } p \text{ and } q \text{ are even} \\ 0 & \text{else} \end{cases}$$

So if we take $p = 1$ and q and r positive odd integers, we get an infinite family of examples with $u(\theta(p, q, r)) = 1$, and $|\tilde{\sigma}(\theta(p, q, r))| = 8(1 + q + r) > 1$.

6. APPENDIX

Here we aggregate information about the behavior of linking number, nullity, component count, signature, and seam vertex count under reversal, mirroring and connected sum.

invariant	mirror	reverse
λ	$\lambda(m(\vec{\Gamma})) = -\lambda(\vec{\Gamma})$	$\lambda(-\vec{\Gamma}) = \lambda(\vec{\Gamma})$
β	$\beta(m(\vec{\Gamma})) = \beta(\vec{\Gamma})$	$\beta(-\vec{\Gamma}) = \beta(\vec{\Gamma})$
μ	$\mu(m(\vec{\Gamma})) = \mu(\vec{\Gamma})$	$\mu(-\vec{\Gamma}) = \mu(\vec{\Gamma})$
σ	$\sigma(m\vec{\Gamma}) = -\sigma(\vec{\Gamma})$	$\sigma(-\vec{\Gamma}) = \sigma(\vec{\Gamma})$
sv	$\mathrm{sv}(m\vec{\Gamma}) = \mathrm{sv}(\vec{\Gamma})$	$\mathrm{sv}(-\vec{\Gamma}) = -\mathrm{sv}(\vec{\Gamma})$

TABLE 1. The behavior of linking number, nullity, component count, signature, and signed seam vertex count under reversal and mirroring, from Lemma 2.24, Definition 3.8, and Lemma 4.4.

invariant	$\#_2$	$\#_3$
β	$\beta(\Gamma_1 \#_2 \Gamma_2) = \beta(\Gamma_1) + \beta(\Gamma_2) + 1$	$\beta(\Gamma_1 \#_3 \Gamma_2) = \beta(\Gamma_1) + \beta(\Gamma_2)$
μ	$\mu(\Gamma_1 \#_2 \Gamma_2) = \mu(\Gamma_1) + \mu(\Gamma_2) - 2$	$\mu(\Gamma_1 \#_3 \Gamma_2) = \mu(\Gamma_1) + \mu(\Gamma_2) - 3$
σ		$\sigma(\vec{\Gamma_1} \#_3 \vec{\Gamma_2}) = \sigma(\vec{\Gamma_1}) + \sigma(\vec{\Gamma_2})$
sv	$\mathrm{sv}(\vec{\Gamma_1} \#_2 \vec{\Gamma_2}) = \mathrm{sv}(\vec{\Gamma_1}) + \mathrm{sv}(\vec{\Gamma_2})$	$\mathrm{sv}(\vec{\Gamma_1} \#_3 \vec{\Gamma_2}) = \mathrm{sv}(\vec{\Gamma_1}) + \mathrm{sv}(\vec{\Gamma_2})$

TABLE 2. The behavior of nullity, component count, signature, and signed seam vertex count under connected sum from Lemmas 2.25, Lemma 3.9, and Lemma 4.4. Signature under order two connected sum is dependent on edge choice and is omitted.

REFERENCES

[BBM+22] Kenneth L. Baker, Dorothy Buck, Allison H. Moore, Danielle O'Donnol, and Scott Taylor. Primality of theta-curves with proper rational tangle unknotting number one. arXiv:2201.08213 [math.GT], 2022.

[BO18] Dorothy Buck and Danielle O'Donnol. Unknotting numbers for prime θ-curves up to seven crossings. arXiv:1710.05237v2 [math.GT], 2018.

[BZH14] Gerhard Burde, Heiner Zieschang, and Michael Heusener. *Knots*, volume 5 of *De Gruyter Studies in Mathematics*. De Gruyter, Berlin, extended edition, 2014.

[CF08] David Cimasoni and Vincent Florens. Generalized Seifert surfaces and signatures of colored links. *Trans. Amer. Math. Soc.*, 360(3):1223–1264, 2008.

[CHK00] Daryl Cooper, Craig D. Hodgson, and Steven P. Kerckhoff. *Three-dimensional orbifolds and cone-manifolds*, volume 5 of *MSJ Memoirs*. Mathematical Society of Japan, Tokyo, 2000. With a postface by Sadayoshi Kojima.

[GL78] C. McA. Gordon and R. A. Litherland. On the signature of a link. *Invent. Math.*, 47(1):53–69, 1978.

[GR18] Catherine Gille and Louis-Hadrien Robert. A signature invariant for knotted Klein graphs. *Algebr. Geom. Topol.*, 18(6):3719–3747, 2018.

[Kin58] Shin'ichi Kinoshita. Alexander polynomials as isotopy invariants. I. *Osaka Math. J.*, 10:263–271, 1958.

[KR21]   Mikhail Khovanov and Louis-Hadrien Robert. Foam evaluation and Kronheimer-Mrowka theories. *Adv. Math.*, 376:Paper No. 107433, 59, 2021.

[KT76]   Louis H. Kauffman and Laurence R. Taylor. Signature of links. *Trans. Amer. Math. Soc.*, 216:351–365, 1976.

[Mas69] W. K. Mason. Homeomorphic continuous curves in 2-space are isotopic in 3-space. *Trans. Amer. Math. Soc.*, 142:269–290, 1969.

[Mur65] Kunio Murasugi. On a certain numerical invariant of link types. *Trans. Amer. Math. Soc.*, 117:387–422, 1965.

[OSB18] Danielle O'Donnol, Andrzej Stasiak, and Dorothy Buck. Two convergent pathways of DNA knotting in replicating dna molecules as revealed by θ-curve analysis. *Nucleic Acids Res*, 46(17):9181–9188, Sep 2018.

[Pow17] Mark Powell. The four-genus of a link, levine–tristram signatures and satellites. *Journal of Knot Theory and Its Ramifications*, 26(02):1740008, 2017.

[Wol87] Keith Wolcott. The knotting of theta curves and other graphs in S^3. In *Geometry and topology (Athens, Ga., 1985)*, volume 105 of *Lecture Notes in Pure and Appl. Math.*, pages 325–346. Dekker, New York, 1987.

Department of Mathematics, University of Miami
Email address: `k.baker@math.miami.edu`

Department of Mathematics & Applied Mathematics, Virginia Commonwealth University
Email address: `moorea14@vcu.edu`

School of Science, Mathematics, and Engineering, Marymount University
Email address: `dodonnol@marymount.edu`

Department of Mathematics, Colby College
Email address: `scott.taylor@colby.edu`