Revolutionaries and spies: Spy-good and spy-bad graphs

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October 11, 2011

Abstract

We study a game played on a graph G by a team of r *revolutionaries* and a team of s *spies*. Initially, revolutionaries and then spies take positions at vertices. In each subsequent round, each revolutionary may move to an adjacent vertex or not move, and then each spy has the same option. The revolutionaries win by holding a meeting of m revolutionaries at some vertex having no spy at the end of a round; the spies win if they can prevent this forever.

Let $\sigma(G, m, r)$ denote the minimum number of spies needed to win. Trivially, $\min\{\lfloor r/m \rfloor, |V(G)|\} \le \sigma(G, m, r) \le r - m + 1$. We prove that $\sigma(G, m, r)$ equals the lower bound whenever G has a rooted spanning tree T such that every edge of G not in T joins two vertices having the same parent in T. Such graphs include graphs that have a dominating vertex. In general, $\sigma(G, m, r) \le \gamma(G) \lfloor r/m \rfloor$, where $\gamma(G)$ is the domination number, and this bound is nearly sharp when $\gamma(G) \le m$.

For fixed r and m, there are chordal graphs and bipartite graphs such that $\sigma(G, m, r) = r - m + 1$, and this holds also for the random graph. Also $\sigma(G, m, r) = r - m + 1$ for hypercubes of dimension at least r when $m = 2$. For $r \ge m \ge 3$, the number of spies needed to win on a hypercube of dimension at least r exceeds $r - \frac{3}{4}m^2$.

For complete k-partite graphs with partite sets of size at least $2r$, the leading term in the threshold for spies to win is approximately $\frac{k}{k-1}\frac{r}{m}$ when $k \ge m$. If G is a complete bipartite graph with such large partite sets, then $\sigma(G, 2, r) = \left\lceil \frac{\lfloor 7r/2 \rfloor - 3}{5} \right\rceil$ and $\sigma(G, 3, r) = \lfloor r/2 \rfloor$. For larger m, the threshold is between $\frac{3r}{2m} - 3$ and $\frac{(1+1/\sqrt{3})r}{m}$.

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1 Introduction

We study a pursuit game between two teams on a graph; it can be viewed as modeling a problem of network security. The first team consists of r *revolutionaries*; the second consists of s *spies*. The revolutionaries want to arrange a one-time meeting of m revolutionaries free of oversight by spies. Initially, the revolutionaries take positions at vertices, and then the spies do the same. In each subsequent round, each revolutionary may move to an adjacent vertex or not move, and then each spy has the same option. Everyone knows where everyone is.

The revolutionaries win if at the end of a round there is an *unguarded meeting*, where a *meeting* is a set of (at least) m revolutionaries on one vertex, and a meeting is *unguarded* if there is no spy at that vertex. The spies win if they can prevent this forever. Let $\mathrm{RS}(G, m, r, s)$ denote this game played on the graph G by s spies and r revolutionaries seeking an unguarded meeting of size m.

The spies trivially win if $s \ge |V(G)|$. The revolutionaries can form $\lfloor r/m \rfloor$ meetings initially; if $s < \lfloor r/m \rfloor$ and $s < |V(G)|$, then the spies immediately lose. On the other hand, the spies win if $s \ge r - m + 1$; they follow $r - m + 1$ distinct revolutionaries, and the other $m - 1$ revolutionaries cannot form a meeting. For fixed G, r, m, let $\sigma(G, m, r)$ denote the minimum s such that the spies win $\mathrm{RS}(G, m, r, s)$.

The game of revolutionaries and spies was invented by Jozef Beck in the mid-1990s. Smyth promptly showed that $\sigma(G, m, r) = \lfloor r/m \rfloor$ when G is a tree, achieving the trivial lower bound (a proof appears in [2]). Howard and Smyth [3] studied the game when G is the infinite 2-dimensional integer grid with one-step horizontal, vertical, and diagonal edges. They observed that the spy wins $\mathrm{RS}(G, m, 2m-1, 1)$ (the spy stays at the median position), and hence $\sigma(G, m, r) \le r - 2m + 2$ for general r and m (note that always $\sigma(G, m, r) \le \sigma(G, m, r-1) + 1$). For $m = 2$, they proved that $6\lfloor r/8 \rfloor \le \sigma(G, 2, r) \le r - 2$; they conjectured that the upper bound is the correct answer.

Cranston, Smyth, and West [2] showed that $\sigma(G, m, r) \le \lceil r/m \rceil$ when G has at most one cycle. Furthermore, let G be a unicyclic graph containing a cycle of length ℓ and t vertices not on the cycle, where $\ell + t > r/m$ and $|V(G)| > r/m$. They showed that if $m \nmid r$, then $\sigma(G, m, r) = \lfloor r/m \rfloor$ if and only if $\ell \le \max\{s - t + 2, 3\}$.

Say that G is *spy-good* if $\sigma(G, m, r)$ equals the trivial lower bound $\lfloor r/m \rfloor$ for all m and r such that $r/m < |V(G)|$. In Section 2, we obtain a large class of spy-good graphs. A *webbed tree* is a graph G containing a rooted spanning tree T such that every edge of G not in T joins vertices having the same parent in T. We prove that every webbed tree is spy-good.

Every graph having a dominating vertex u is a webbed tree (rooted at u). The upper bound for such graphs generalizes: always $\sigma(G, m, r) \le \gamma(G)\lfloor r/m \rfloor$, where $\gamma(G)$ is the domination number of G (the minimum size of a set S such that every vertex outside S has a neighbor in S). Since always $\lfloor r/m \rfloor \ge (r - m + 1)/m$, this upper bound is nontrivial only when $\gamma(G) \le m$. In that case, it is nearly sharp: for $t, m, r \in \mathbb{N}$ with $t \le m$, we construct a graph with domination number t such that $\sigma(G, m, r) > t(r/m - 1)$.

In contrast to spy-good graphs, we say that a graph G is *spy-bad* (for r revolutionaries and meeting size m) if $r - m$ spies cannot win, so $\sigma(G, m, r) = r - m + 1$ and the trivial upper bound is sharp. We consider such graphs in Section 3. Some chordal graphs (and bipartite graphs) are spy-bad (for particular r and m), and the random graph is almost surely spy-bad.

We also study hypercubes in Section 3, showing first that the d-dimensional hypercube Q_d is spy-bad when $d \geq r$ and $m = 2$. Indeed, the subgraph of Q_d consisting of the vertices within distance k of a fixed vertex has the same property when $k \geq 2$. Also, if $d < r \leq 2^d/d^8$, then the revolutionaries can beat $(d-1)\lfloor r/d \rfloor$ spies on Q_d (for $m = 2$). For general m, we show that hypercubes are nearly spy-bad by proving that $\sigma(Q_d, m, r) \geq r - \frac{3}{4}m^2$ when $d \geq r \geq m$. Possibly the revolutionaries also win against $r - cm$ spies for some constant c.

In Section 4, we consider complete k-partite graphs. A complete k-partite graph is r-*large* if each part has at least $2r$ vertices, which is as many vertices as the players might want to use. For large k, such graphs are "nearly" spy-good: $\sigma(G, m, r) \geq \frac{k}{k-1}\frac{r}{m} + k$. Also $\sigma(G, m, r) \geq \frac{k}{k-1}\frac{r}{m+c} - k$ when $k \geq m$ and $c = \frac{1}{k-1}$.

Section 5 focuses on complete bipartite graphs and contains our most intricate results. When G is an r-large complete bipartite graph, we obtain $\sigma(G, 2, r) = \left\lceil \frac{\lfloor 7r/2 \rfloor - 3}{5} \right\rceil$ and $\sigma(G, 3, r) = \lfloor r/2 \rfloor$. For larger m we do not have the complete answer; we prove

$$\left(\frac{3}{2} - o(1)\right)\frac{r}{m} - 2 \leq \sigma(G, m, r) \leq \left(1 + \frac{1}{\sqrt{3}}\right)\frac{r}{m} < 1.58\frac{r}{m},$$

where the upper bound requires $\frac{r}{m} \geq \frac{1}{1 - 1/\sqrt{3}}$. We conjecture that $\sigma(G, m, r) = \frac{3r}{2m}$ when 3 divides m, but in other cases the revolutionaries do a bit better. That advantage should fade as m grows, with $\sigma(G, m, r) \sim \frac{3r}{2m}$.

Upper bounds for $\sigma(G, m, r)$ are proved by giving strategies for the spies, typically to maintain certain invariants that keep the meetings guarded. Lower bounds are proved by strategies for the revolutionaries. One may wonder how many rounds the revolutionaries need to win when they can win. Most of our winning strategies for revolutionaries take at most two rounds, but on hypercubes they take $m - 1$ rounds. In [2], strategies for revolutionaries proving that $\sigma(C_n, m, r) = \lceil r/m \rceil$ (when $r/m > n$) may take many rounds.

It would be interesting to characterize spy-good graphs. In all known spy-good graphs, the spies can ensure that at the end of each round the number of spies on any vertex v is at least $\lfloor r(v)/m \rfloor$, where $r(v)$ is the number of revolutionaries at v. Existence of such a strategy is preserved when any vertex expands into a complete subgraph. Also, Howard and Smyth [3] observed that $\sigma(G, m, r)$ is preserved by taking the distance power of a graph. Hence every graph obtained from some webbed tree via some sequence of distance powers or vertex expansions is spy-good. Are these the only spy-good graphs?

2 Spy-Good Graphs

We begin with graphs having a *dominating vertex* (a vertex adjacent to all others); later we apply this result to webbed trees. Let $N(v)$ denote the neighborhood of a vertex v. Also $N[v] = N(v) \cup \{v\}$, and $N(S) = \bigcup_{v \in S} N(v)$.

Definition 2.1. For a graph G having a dominating vertex u, the position at the end of a round In the game $\mathrm{RS}(G, m, r, s)$ is *stable* if, for each vertex v other than u, the number of spies at v is exactly $\lfloor r(v)/m \rfloor$, where $r(v)$ is the number of revolutionaries at v. The remaining spies, if any, are at u.

Theorem 2.2. *If a graph G has a dominating vertex, then $\sigma(G, m, r) = \lfloor r/m \rfloor$.*

Proof. Let u be a dominating vertex in G, and let $s = \lfloor r/m \rfloor$. Since $s = \lfloor r/m \rfloor$, a stable position will have a spy at u if there is a meeting at u. Hence a stable position has no unguarded meeting. When $s = \lfloor r/m \rfloor$, there are enough spies to achieve a stable position after the initial round. We give a strategy for the spies to achieve a stable position at the end of each round and hence win.

Suppose the position is stable after round t. Let X be a maximal family of disjoint sets of m revolutionaries on vertices other than u at the end of round t. Let Y be such a maximal family after the revolutionaries move in round $t+1$. Let $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots y_{k'}\}$. In X or Y, more than one set may correspond to a single vertex in G. For example, a vertex v having $pm + q$ revolutionaries at the end of round t corresponds to p elements of X, and there are p spies at v after round t. Let $X' = \{x_{k+1}, \ldots, x_s\}$, representing the excess spies waiting at u after round t.

Define an auxiliary bipartite graph H with partite sets $X \cup X'$ and Y. For $x_i \in X$ and $y_j \in Y$, put $x_i y_j \in E(H)$ if some revolutionary from meeting x_i is in meeting y_j (note that x_i and y_j may be the same set). Also make all of X' adjacent to all of Y. If some matching in H covers Y, then the spies can move so that every vertex other than u having $p'm + q'$ revolutionaries after round $t+1$ has exactly p' spies on it (and the remaining spies are at u).

The existence of such a matching follows from Hall's Theorem. For $S \subseteq Y$, always $X' \subseteq N(S)$, so $|N(S)| = |X'| + |N(S) \cap X|$. Consider the $m|S|$ revolutionaries in the meetings corresponding to S. Such revolutionaries came from meetings in $|N(S) \cap X|$ or were not in any of the k meetings indexed by X. Hence $m|S| \le m|N(S) \cap X| + (r - km)$. Since $|X'| = s - k$ and $s = \lfloor r/m \rfloor$,

$$|N(S)| \ge |X'| + |S| - (\lfloor r/m \rfloor - k) = s - k + |S| - (\lfloor r/m \rfloor - k) = |S|,$$

so Hall's Condition holds. □

Corollary 2.3. *Fix n, m, r with $n \ge r/m$. For $0 \le k \le \binom{n}{2}$, there is an n-vertex graph G with k edges such that $\sigma(G, m, r) = \lfloor r/m \rfloor$.*

Proof. For $k \geq n$, form G by adding the desired number of edges joining leaves of an n-vertex star; Theorem 2.2 applies. For $k \leq n-1$, let G be a star plus isolated vertices; use Theorem 2.2 and $\lfloor a \rfloor + \lfloor b \rfloor \leq \lfloor a+b \rfloor$. $\square$

Corollary 2.4. *For any graph G, always $\sigma(G,m,r) \leq \gamma(G)\lfloor r/m \rfloor$, where $\gamma(G)$ is the domination number of G.*

Proof. Let S be a smallest dominating set. With each vertex $u \in S$, associate $\lfloor r/m \rfloor$ spies. Let G_u be the subgraph of G induced by $N[u]$; it has u as a dominating vertex. The spies associated with u stay in G_u, following the strategy of Theorem 2.2 on G_u. When there are fewer than r revolutionaries in G_u, the spies imagine that the missing ones are at u. When a real revolutionary comes to vertex v in G_u from outside G_u, a revolutionary in the imagined game moves from u to v to perform its moves. When the real revolutionary leaves G_u, the revolutionary tracking it in the game on G_u returns to u. These moves are possible, since u is a dominating vertex in G_u. Since the spies win each imagined game, the revolutionaries in the real game never make an unguarded meeting at the end of a round. $\square$

As remarked in the introduction, Corollary 2.4 is of interest only when $\gamma(G) \leq m$, because otherwise the trivial upper bound $r-m+1$ is stronger. When $\gamma(G) \leq m$, the bound in Corollary 2.4 cannot be improved. Since this is proved by giving a strategy for revolutionaries, we postpone it to the next section. Meanwhile, we extend the result of Theorem 2.2 to a more general class of graphs.

Definition 2.5. For any vertex v in a rooted tree, the *parent* of a non-root vertex v (written v^+) is the first vertex after v on the path from v to the root. The set of *children* of v (written $C(v)$) is the set of neighbors of v other than its parent, and the set of *descendants* of v (written $D(v)$) is the set of vertices whose path to the root contains v.

A *webbed tree* is a graph G having a rooted spanning tree T such that every edge of G outside T joins two vertices having the same parent (called *siblings*). In a graph with a dominating vertex, *stabilization* is the process of spies reestablishing a stable position after a move by the revolutionaries from a stable position.

Trivially, all trees and all graphs having a dominating vertex are webbed trees. A 2-connected graph is a webbed tree if and only if it has a dominating vertex. Every webbed tree is a graph whose blocks have dominating vertices, but the converse does not hold. Consider the graph obtained from two 4-cycles with a common vertex by adding chords of the 4-cycles to create four vertices of degree 3; every block has a dominating vertex, but the graph is not a webbed tree.

Our main result in this section is that all webbed trees are spy-good. The argument includes a proof of the result in [2] for trees. When G is a tree, choosing any vertex as a root expresses G as a webbed tree. The strategy for spies on rooted trees in [2] maintains an invariant ensuring that all meetings are guarded. We use the same invariant for webbed trees, but in this more general class the strategy to maintain it is more subtle. After the revolutionaries move, the spies restore the invariant by applying the strategy in Theorem 2.2 independently to each graph induced by a vertex and its children in the spanning tree.

Theorem 2.6. *If G is a webbed tree, then $\sigma(G,m,r) = \lfloor r/m \rfloor$.*

Proof. Let T be a rooted spanning tree in G such that every edge of G not in T joins sibling vertices in T. Let z be the root of T, and let $s = \lfloor r/m \rfloor$. The notation for children and descendants is as in Definition 2.5 with respect to T.

For each vertex v, let $r(v)$ and $s(v)$ denote the number of revolutionaries and spies on v at the current time, respectively, and let $w(v) = \sum_{u \in D(v)} r(u)$. The spies maintain the following invariant specifying the number of spies on each vertex at the end of any round:

$$s(v) = \left\lfloor \frac{w(v)}{m} \right\rfloor - \sum_{x \in C(v)} \left\lfloor \frac{w(x)}{m} \right\rfloor \qquad \text{for } v \in V(G). \tag{1}$$

Since $\sum_{x \in C(v)} w(x) = w(v) - r(v)$, the formula is always nonnegative. Also, if $r(v) \geq m$, then $s(v) \geq \left\lfloor \frac{w(v)}{m} \right\rfloor - \left\lfloor \frac{w(v)-r(v)}{m} \right\rfloor \geq 1$. Hence (1) guarantees that every meeting is guarded.

To show that the spies can establish (1) after the first round, it suffices that all the formulas sum to $\lfloor r/m \rfloor$. More generally, summing over the descendants of any vertex v,

$$\sum_{u \in D(v)} s(u) = \left\lfloor \frac{w(v)}{m} \right\rfloor, \tag{2}$$

since $\lfloor w(u)/m \rfloor$ occurs positively in the term for u and negatively in the term for u^+, except that $\lfloor w(v)/m \rfloor$ occurs only positively. When $v = z$, the total is $\lfloor r/m \rfloor$, since $w(z) = r$.

To show that the spies can maintain (1), let $r'(v)$ denote the new number of revolutionaries at v after the revolutionaries move, and let $w'(v) = \sum_{u \in D(v)} r'(v)$. The spies will move to achieve the new values required by (1). To determine these moves, we will use stabilization on each subgraph induced by a vertex and its children, independently. Let $G(v)$ denote the subgraph induced by $C(v) \cup \{v\}$; note that v is a dominating vertex in $G(v)$. We will play a round in an imagined "local" game on each $G(v)$.

To set up the local games, we partition the $s(v)$ spies at each vertex v into a set of $\check{s}(v)$ spies to be used in the local game on $G(v)$ and a set of $\hat{s}(v)$ spies to be used in the local game on $G(v^+)$, where $\check{s}(v)$ and $\hat{s}(v)$ sum to $s(v)$ (when the tree is drawn with the root z at the top, the accent indicates the direction of the relevant subgraph).

Let $D^*(v) = D(v) - \{v\}$. Let $w^*(v)$ be the number of revolutionaries that are in $D^*(v)$ at the end of the previous round *or* are there after the revolutionaries move in the new round. Every revolutionary counted by $w^*(v)$ is also counted by $w(v)$, and every revolutionary counted by $\sum_{x\in C(v)} w(x)$ is also counted by $w^*(v)$. These statements also hold with w' in place of w. Hence

$$w(v) \geq w^*(v) \qquad \text{and} \qquad w^*(v) \geq \sum_{x\in C(v)} w(x). \tag{3}$$

By (3), $\hat{s}(v)$ and $\check{s}(v)$ are nonnegative when we define

$$\hat{s}(v) = \left\lfloor \frac{w(v)}{m} \right\rfloor - \left\lfloor \frac{w^*(v)}{m} \right\rfloor \qquad \text{and} \qquad \check{s}(v) = \left\lfloor \frac{w^*(v)}{m} \right\rfloor - \sum_{x\in C(v)} \left\lfloor \frac{w(x)}{m} \right\rfloor. \tag{4}$$

By (1), $\hat{s}(v) + \check{s}(v) = s(v)$. Note also that if v is a leaf of T, then $\check{s}(v) = 0$ and $\hat{s}(v) = s(v)$.

For each non-leaf vertex v, the spies first imagine positions of revolutionaries in a game on the graph $G(v)$ that together with (4) for the spies form a stable position. After viewing the actual moves by revolutionaries within $G(v)$ as moves in this game, the spies stabilize as in Theorem 2.2. We will show that the resulting positions of spies satisfy the global invariant. The spies imagine $\hat{r}(v)$ spies at v in $G(v^+)$ and $\check{r}(v)$ spies at v in $G(v)$, where

$$\hat{r}(v) = w(v) - m\left\lfloor \frac{w^*(v)}{m} \right\rfloor \qquad \text{and} \qquad \check{r}(v) = w^*(v) - \sum_{x\in C(v)} w(x). \tag{5}$$

By (3), the values of $\check{r}(v)$ and $\hat{r}(v)$ are nonnegative. Furthermore, we claim that (4) and (5) together define a stable position. In $G(v)$ we use $\check{s}(v)$ and $\check{r}(v)$, and we use $\hat{s}(x)$ and $\hat{r}(x)$ for $x \in C(v)$. By definition, $\hat{s}(x) = \lfloor \hat{r}(x)/m \rfloor$. It remains only to check the sum. We compute the total number of revolutionaries in the local game:

$$\check{r}(v) + \sum_{x\in C(v)} \hat{r}(x) = w^*(v) - \sum_{x\in C(v)} w(x) + \sum_{x\in C(v)} w(x) - m \sum_{x\in C(v)} \left\lfloor \frac{w^*(x)}{m} \right\rfloor$$

Dividing by m yields $\frac{w^*(v)}{m} - \sum_{x\in C(v)} \left\lfloor \frac{w^*(x)}{m} \right\rfloor$, whose floor is $\check{s}(v) + \sum_{x\in C(v)} \hat{s}(x)$, as desired.

Before stabilizing, the spies also imagine moves by revolutionaries in the game on $G(v)$. In fact, they use the actual moves by revolutionaries in the global game. Each such move occurs within the subgraph $G(v)$ for one local game. The local game can model these moves if the relevant value of $\hat{r}$ or $\check{r}$ is at least the number of real revolutionaries leaving this vertex and staying within this subgraph. The revolutionaries leaving v by edges in $G(v^+)$ are those that were in $D(v)$ and now are not; there are at most $w(v) - w^*(v)$. By (5), $\hat{r}(v)$ is at least this large. Similarly, revolutionaries leaving v via $G(v)$ wind up in $D^*(v)$ but were not there previously, so the number of them is at most $w^*(v) - \sum_{x\in C(v)} w(x)$, which equals $\check{r}(v)$.

The net change in the actual number of revolutionaries at v is $r'(v) - r(v)$. Some of this change is due to moves in $G(v)$ and the rest to moves in $G(v^+)$. Moves in $G(v^+)$ enter or leave $D(v)$. Hence the net change in the number of revolutionaries at v due to such moves is $w'(v) - w(v)$. The remaining net change, due to moves between v and its children (in

$G(v)$), is $(r'(v) - r(v)) - (w'(v) - w(v))$. Therefore, after executing the actual moves in the imagined local games, the new imagined distributions for the revolutionaries are given by

$$\hat{r}'(v) = \hat{r}(v) + w'(v) - w(v) \quad \text{and} \quad \check{r}'(v) = \check{r}(v) + (r'(v) - r(v)) - (w'(v) - w(v)). \tag{6}$$

The specification of $\hat{r}(v)$ in (5) and the change from $\hat{r}(v)$ to $\hat{r}'(v)$ in (6) immediately yield the formula for $\hat{r}'(v)$ in (7). To obtain $\check{r}'(v)$, start with the formula for $\check{r}'(v)$ in (5) and adjust by the definitions of $r(v) - r(v)$ and $w'(v) - r'(v)$, as indicated in (6). We compute

$$\begin{aligned}\check{r}'(v) &= \check{r}(v) + (w(v) - r(v)) - (w'(v) - r'(v)) \\ &= w^*(v) - \sum_{x \in C(v)} w(x) + \sum_{x \in C(v)} w(x) - \sum_{x \in C(v)} w'(x) = w^*(v) - \sum_{x \in C(v)} w'(x).\end{aligned}$$

Thus

$$\hat{r}'(v) = w'(v) - m\left\lfloor \frac{w^*(v)}{m} \right\rfloor \qquad \text{and} \qquad \check{r}'(v) = w^*(v) - \sum_{x \in C(v)} w'(x). \tag{7}$$

Applying stabilization yields new spy distributions on the local games. By Theorem 2.2, these positions are stable, so $\hat{s}'(x) = \lfloor \hat{r}'(x)/m \rfloor$ for $x \in C(v)$, and $\check{s}'(v)$ is the leftover amount for v in the local game on $G(v)$. By the same computation that earlier showed $\check{s}(v)$ was the correct needed amount of spies left for v in $G(v)$, also $\check{s}'(v) = \left\lfloor \frac{w^*(v)}{m} \right\rfloor - \sum_{x \in C(v)} \left\lfloor \frac{w'(x)}{m} \right\rfloor$.

Because each spy participated in exactly one local game, playing the local games independently ensures automatically that each spy moves along at most one edge. Hence the spy moves we have described are feasible. It remains only to show that (1) holds for the resulting distribution of spies; that is

$$\hat{s}'(v) + \check{s}'(v) = \left\lfloor \frac{w'(v)}{m} \right\rfloor - \sum_{x \in C(v)} \left\lfloor \frac{w'(x)}{m} \right\rfloor \qquad \text{for } v \in V(G).$$

Since the terms involving w^* again cancel, we use (7) to show that $\hat{s}'(v) + \check{s}'(v)$ equals the desired value $s'(v)$ in the same way we used (5) to show that the invented values $\hat{s}(v)$ and $\check{s}(v)$ sum to $s(v)$. □

Beyond webbed trees, consider cycles: $\sigma(C_n, m, r) = \lceil r/m \rceil$ when $n \ge r/m$ [2], so cycles are not spy-good. In a unicyclic graph, the value of σ is $\lceil r/m \rceil$ or $\lfloor r/m \rfloor$, depending on the relationship between $\lfloor r/m \rfloor$ and the number of vertices outside the cycle [2]. Such graphs are not spy-good, because "spy-good" requires $\sigma(G, m, r) = \lfloor r/m \rfloor$ for all r and m.

It is not true that all spy-good graphs are webbed trees. Let G^k denote the kth distance power of G; that is, $V(G^k) = V(G)$, and $E(G^k) = \{uv\colon d_G(u, v) \le k\}$. The spies can simulate one round of the game on G^k by playing k rounds on G. Thus $\sigma(G^k, m, r) \le \sigma(G, m, r)$, as noted by Howard and Smyth [3]. This makes the square of a webbed tree spy-good, even though it is not generally a webbed tree (consider $G = P_n$, for example).

Say that a spy strategy is *conformal* if at the end of each round the number of spies at each vertex v is at least $\lfloor r(v)/m \rfloor$, where $r(v)$ is the number of revolutionaries there. For any conformal spy strategy on G, the strategy described above for G^k is also conformal. Another graph operation also preserves the existence of conformal strategies.

Proposition 2.7. *Obtain G' from a graph G by expanding a vertex of G into a clique. If $\lfloor r/m \rfloor$ spies win $\mathrm{RS}(G,m,r,s)$ by a conformal strategy, then the same holds for G'.*

Proof. Let Q be the clique into which vertex v of G is expanded to form G'. The spies play on G' by imagining a game on G. At each round, the revolutionaries on Q in G' are collected onto v in G, with $r(v)$ there after the previous round and $r'(v)$ after the revolutionaries move. For other vertices, the amounts before and after are as in the real game on G'.

Since $\sum \lfloor a_i \rfloor \le \lfloor \sum a_i \rfloor$, the spies on v at the end of the round in G suffice to cover the $r'(v)$ revolutionaries on Q in G and can move there, since all vertices of Q have the same neighbors outside Q that v has in G. Extra spies move to any vertex of Q. Movements of spies from v in G can also be matched by moves in the game on G'. Other movements are the same in G and G'. This produces a conformal strategy on G'. □

Proposition 2.8. *On a webbed tree G, the winning strategy in Theorem 2.6 is conformal.*

Proof. Let T be a rooted spanning tree such that edges outside T join siblings in T. After each round, the number of spies on vertex v is given by

$$\left\lfloor \frac{r(v) + \sum_{x \in C(v)} w(x)}{m} \right\rfloor - \sum_{x \in C(v)} \left\lfloor \frac{w(x)}{m} \right\rfloor .$$

Since $\sum \lfloor a_i \rfloor \le \lfloor \sum a_i \rfloor$, the strategy is conformal. □

These results imply that graphs obtained from webbed trees by vertex expansions and distance powers are spy-good. For example, the square of a path is spy-good. This graph is not a webbed tree, since it is 2-connected but has no dominating vertex (when it has at least six vertices). On the other hand, it is an interval graph, where an *interval graph* is a graph representable by assigning each vertex v an interval on the real line so that vertices are adjacent if and only if their intervals intersect. An interval graph that is not a distance power and has no two vertices with the same closed neighborhood is obtained from the square of an 8-vertex path by adding on edge joining the third and sixth vertices.

Question 2.9. *Do there exist spy-good graphs other than the graphs obtained from webbed trees by any combination of vertex expansions and distance powers? In particular, is every interval graph spy-good?*

If interval graphs are not all spy-good, they may still satisfy a nice upper bound on $\sigma(G,m,r)$ in terms of r/m, like cycles do.

3 Spy-Bad Graphs

Going beyond interval graphs to chordal graphs introduces graphs that need many spies to prevent meetings. A *split graph* is a graph whose vertices can be partitioned into a clique and an independent set. A *chordal graph* is a graph in which every cycle of length at least 4 has a chord; split graphs clearly have this property. Recall that for fixed r and m a graph is spy-bad if the revolutionaries can beat $r-m$ spies ($r-m+1$ spies trivially win).

Proposition 3.1. *Given $r, m \in \mathbb{N}$, there is a chordal graph G (in fact a split graph) such that $\sigma(G,m,r) = r-m+1$.*

Proof. Let $G_{m,r}$ be the split graph consisting of a clique Q of size r and an independent set S of size $\binom{r}{m}$, with the neighborhoods of the vertices in S being distinct m-sets in Q. We show that $r-m$ spies cannot win.

The revolutionaries initially occupy each vertex of Q. Let s' be the number of vertices of Q initially occupied by spies. The number of threatened meetings that spies on Q are not adjacent to is $\binom{r-s'}{m}$. Protecting against such threats requires putting spies initially on the $\binom{r-s'}{m}$ vertices of S corresponding to these m-sets, but only $r-m-s'$ remaining spies are available, and $\binom{r-s'}{m} > r-m-s'$ when $r-s' \geq m$. □

Note that $\frac{r-m+1}{r/m}$ can be made arbitrarily large. When $r = 2m$, the ratio exceeds $m/2$. Letting m also grow, we observe that $\sigma(G,m,r)$ cannot be bounded by a constant multiple of r/m, even on split graphs. Furthermore, the strategy for revolutionaries in Proposition 3.1 does not use any edges within the clique, so the statement remains true also for the bipartite graph obtained by deleting those edges.

A similar construction allows us to show that Corollary 2.4 is nearly sharp. When $t = m$, the upper and lower bounds are equal; when $m \mid r$, the difference between them is $t-1$.

Theorem 3.2. *Given $t, m, r \in \mathbb{N}$ such that $t \leq m \leq r-m$, there is a graph G with domination number t such that $\sigma(G,m,r) > t(r/m-1)$.*

Proof. First we construct a graph G. Begin with a copy of $K_{t,r}$ having partite sets T of size t and R of size r. Add an independent set U of size $t\binom{r}{m}$, grouped into sets of size t. With each m-set A in R, associate one t-set A' in U. Make all of A adjacent to all of A', and add a matching joining A' to T. Note that T is a dominating set.

To show that $\gamma(G) = t$, note that $t \leq m \leq r-m$. Let S be a smallest dominating set. For each m-set A in R, there are t vertices in U that are adjacent only to A in R. Thus if $|S \cap R| \leq r-m$, then some t-set A' in U is undominated by $S \cap R$. Outside of R, the closed neighborhoods of the vertices in A' are pairwise disjoint, so S needs t additional vertices to dominate them. Hence $\gamma(G) < t$ requires $r-m+1 < t$, but we are given $t \leq r-m$.

Now, we give a strategy for the revolutionaries to win against $t(r/m-1)$ spies on G. Let $s = \lfloor t(r/m-1) \rfloor$. The revolutionaries initially occupy R, one on each vertex. A spy on a vertex u of U can protect all the same threats (and more) by locating at the neighbor of u in T instead. Hence we may assume (at least for the purpose of trying to survive the next round) that no spies locate initially in U.

Let v be a vertex of T having the fewest initial spies, and let $s(v)$ be the number of spies there. The revolutionaries will win by attacking the neighbors of v. Let s' be the number of spies initially in R, so $s(v) \leq (s - s')/t$.

The revolutionaries want to form meetings at $s(v)+1$ neighbors of v that are neighbors of no other vertices with spies. Let R' be the set of vertices in R that do not have spies; note that $|R'| \geq r - s'$. If $|R'| \geq m(s(v)+1)$, then the revolutionaries win as follows. First, group vertices in R' into $s(v)+1$ sets of size m. For each such set A, the revolutionaries on A move to the unique vertex $u_{A,v}$ in the associated subset A' of U that is adjacent to v in T. For each such vertex, the only neighbor having a spy is v, so the meetings cannot all be guarded and the revolutionaries win.

It suffices to show that $r - s' \geq m(s(v)+1)$. Since v has the fewest spies among vertices of T, we have $ts(v) \leq s - s' \leq t(r/m - 1) - s'$. Multiplying by m/t and adding m yields $m(s(v)+1) \leq r - s'(m/t) \leq r - s'$, as desired, using $t \leq m$ at the end. □

Although the construction in Theorem 3.2 depends heavily on m, it does not depend much on r. Indeed, the construction works equally well whenever the number of revolutionaries is at most r, because the revolutionaries can use the strategy for a smaller number of revolutionaries on the appropriate subgraph of the graph constructed for r revolutionaries. The same comment applies to Proposition 3.1.

Next we consider random graphs in the Erdős–Renyi binomial model $G(n,p)$: within the vertex set $[n]$, pairs of vertices occur as edges independently with probability p, and we say that an event occurs *almost surely* if its probability tends to 1 as $n \to \infty$. The random graph is almost surely spy-bad in a rather strong sense: no matter where the revolutionaries start (on distinct vertices), there will almost surely be a vertex that m revolutionaries and no spies can move to.

This outcome follows from an elementary property of random graphs that holds almost surely even when p and r can vary with n in somewhat restricted ways. Motivated by Alon and Spencer [1], say that G has the *r-extension property* if for any disjoint $T, U \subset V(G)$ with $|T| + |U| \leq r$, there is a vertex $x \in V(G)$ adjacent to all of T and none of U. We first say precisely why this property makes the game easy for the revolutionaries.

Proposition 3.3. *If a graph G satisfies the r-extension property, and $m \leq r' \leq r$, then G is spy-bad for r' revolutionaries and meeting size m.*

Proof. The r revolutionaries initially occupy any set of r vertices in G. To see that $r - m$ spies cannot prevent them from winning on the first round, let U be the set occupied by the spies, and let T be the set occupied by uncovered revolutionaries. The revolutionaries on T win by moving to the vertex x guaranteed by the r-extension property. $\square$

Alon and Spencer present the result below for constant r (Theorem 10.4.5 in the third edition), but the proof holds more generally.

Theorem 3.4. *Let $\epsilon = \min\{p, 1-p\}$, where p is a probability that depends on n. If $r = o\left(\frac{n\epsilon^r}{\ln n}\right)$ and $n\epsilon^r \to \infty$, then $G(n,p)$ almost surely has the r-extension property (and hence is spy-bad for all m and r' with $m \le r' \le r$).*

Proof. Let G be distributed as $G(n,p)$. Given $T, U \subset V(G)$ with $|T|+|U| \le r$, write $t = |T|$ and $u = |U|$. For $x \in V(G) - (T \cup U)$, let $A_{T,U,x}$ be the event that x is adjacent to all of T and none of U; note that $\mathbb{P}[A_{T,U,x}] = p^t(1-p)^u \ge \epsilon^r$.

Let $A_{T,U}$ be the event that $A_{T,U,x}$ fails for all $x \in V(G) - (T \cup U)$. The events $A_{T,U,x}$ for different x are determined by disjoint sets of vertex pairs, so $\mathbb{P}[A_{T,U}] \le (1-\epsilon^r)^{n-r} \le e^{-\epsilon^r(n-r)}$.

The r-extension property fails if and only if some event of the form $A_{T,U}$ occurs. Hence it suffices to show that the probability of their union tends to 0. There are 3^r ways to form T and U within a fixed r-set of vertices, since a vertex can be in either set or be omitted, and there are $\binom{n}{r}$ sets of size r. Hence the union consists of at most $(3n)^r$ events, each of whose probability is at most $e^{-\epsilon^r(n-r)}$. We compute

$$(3n)^r e^{-\epsilon^r(n-r)} = e^{r\ln(3n)-\epsilon^r(n-r)} = e^{r\ln 3 + r\ln n - \epsilon^r(n-r)}.$$

Since $\epsilon \le 1/2$, the condition $r = o\left(\frac{n\epsilon^r}{\ln n}\right)$ implies $r = o(n)$, so the expression inside the exponent is dominated by $-n\epsilon^r$ and tends to $-\infty$. Thus the bound on the probability that the r-extension property fails tends to 0, and $G(n,p)$ almost surely satisfies the r-extension property. $\square$

In particular, when p is constant, $G(n,p)$ is almost surely spy-bad whenever $m \le r$ with $r \le c\log_{1/\epsilon} n$ with $c < 1$. Similarly, when r is constant, $G(n,p)$ is almost surely spy-bad when p tends to 0 more slowly than $1/n^{1/r}$. With $p \le 1/2$, the key condition is $np^r \to \infty$.

For our final class of spy-bad graphs, we consider the d-dimensional hypercube Q_d: the vertex set is $\{v_S\colon S \subseteq [d]\}$, where $[d] = \{1, \ldots, d\}$, with v_S and v_T adjacent when the symmetric difference of S and T has size 1. The *weight* of a vertex is the size of the set in its subscript. We will consider only vertices of small weight and hence write the subscripts without set brackets. We show that Q_d is spy-bad for $m = 2$ when $d \ge r$. This is an exact answer for these graphs when $m = 2$; for larger m, we will later obtain a lower bound using the same basic idea.

Theorem 3.5. *If $G = Q_d$ and $d \ge r$, then $\sigma(G, 2, r) = r - 1$.*

Proof. The upper bound is trivial; we show that $r-2$ spies cannot win. The revolutionaries begin by occupying $v_1, \ldots, v_r$, threatening meetings of size 2 at $\varnothing$ and at $\binom{r}{2}$ vertices of weight 2. Let t be the number of revolutionaries left uncovered by the initial placement of the spies. Threats at $\binom{t}{2}$ vertices must be watched by spies not on vertices of weight 1. A spy at a vertex of weight 2 can watch one such threat; spies at vertices of weight 3 can watch three of them. Hence $s \geq (r-t) + \frac{1}{3}\binom{t}{2}$ if the spies stop the revolutionaries from winning on the first round. This yields $s \geq r-1$ if $t \geq 5$ or $t \leq 2$.

If $t = 4$ and $s = r-2$, then the spies need to watch six threats at weight 2 using two spies at vertices of weight 3. A spy at a vertex of weight 3 watches the three pairs in its name. Since the edges of a complete graph with four vertices (corresponding to the uncovered revolutionaries) cannot be covered with two triangles, $r-2$ spies are not enough when $t = 4$.

If $t = 3$, then the counting bound yields $s \geq r-2$ for spies to avoid losing on the first round. If the initial placement of $r-2$ spies can watch all immediate threats, then they must cover $r-3$ revolutionaries at vertices of weight 1 and occupy one vertex at weight 3. By symmetry, we may assume the spies locate at v_{123} and $v_4, \ldots, v_r$.

In the first round, revolutionaries at v_1 and v_2 move to $v_\varnothing$; the others wait where they are. To guard the meeting at $v_\varnothing$, a spy at some vertex of weight 1 must move there; let v_j be the vertex from which a spy moves to $v_\varnothing$.

In the second round, the revolutionaries at v_3 and v_j move to v_{3j}, winning. The distance from each spy to v_{3j} after round 1 is at least 3, except for the spy at v_j, so no other spy could have moved after round 1 to watch that threat. $\square$

Extra spies on vertices of weight at least 5 cannot prevent the revolutionaries from winning with the strategy given in the proof of Theorem 3.5. This enables the revolutionaries to win against somewhat fewer spies when r is larger than the dimension.

A *code* with length d and distance k is a set of vertices in Q_d such that the distance between any two of them is at least k. Let $A(d,k)$ denote the maximum size of a code with distance k in Q_d, and let $B(d,k)$ be the number of vertices with distance less than k from a fixed vertex in Q_d. Note that $B(d,k) = \sum_{i=0}^{k-1}\binom{d}{i} \sim d^{k-1}/(k-1)!$ when k is fixed. If $M < 2^d/B(d,k)$, then any code of size M having distance k can be extended by adding some vertex, so $A(d,k) \geq 2^d/d^{k-1}$.

Corollary 3.6. *If $d < r \leq 2^d/d^8$, then $\sigma(Q_d, 2, r) \geq (d-1)\lfloor r/d \rfloor$.*

Proof. Let X be a code with distance 9 in Q_d. The revolutionaries devote d revolutionaries to playing the strategy in the proof of Theorem 3.5 at each of $\lfloor r/d \rfloor$ vertices of X. If the ball of radius 4 at any such vertex has fewer than $d-1$ spies in the initial configuration, then the revolutionaries win in that ball in two rounds, since any spy initially outside that ball is too far away to guard a meeting formed at distance 2 from the central point in round 2.

Since the code has distance 9, the balls of radius 4 are disjoint. Hence $(d-1)\lfloor r/d \rfloor$ spies are needed to keep the revolutionaries from winning within two rounds. $\square$

Theorem 3.5 and Corollary 3.6 can be combined by saying that r revolutionaries win against $r - \lceil r/d \rceil - 1$ spies on Q_d unless $d < \log_2 r + 8 \log_2 \log_2 r$. This result is not always sharp, since three revolutionaries easily beat one spy on Q_2 by starting initially at distinct vertices. Although four revolutionaries can threaten meetings at all eight vertices of Q_3, two spies can watch all those meetings and survive the next round. It appears that $\sigma(Q_3, 2, 4) = 2$, though we have not worked out a complete strategy for two spies against four revolutionaries. We have no nontrivial general upper bounds on $\sigma(Q_d, 2, r)$ when $r > d$.

Next we consider the game on hypercubes when $m > 2$. The same idea of counting threats made by revolutionaries placed initially at vertices of weight 1 will yield $\sigma(Q_d, m, r) > r - 3m^2/4$ when $d \ge r \ge m \ge 3$. We separate this from the argument for $m = 2$ for several reasons. When $m > 2$, we will discard lower-order terms to simplify the argument, and they cannot be discarded to get the optimal bound for $m = 2$. We also used the threat at $\varnothing$, which we will now ignore. Some of the computations for larger m would be invalid as stated, having $m - 2$ in the denominator. Finally, using the second general lemma would force a weaker lower bound for $m = 2$ than we proved.

We begin by proving two lemmas, valid for $m \ge 2$.

Lemma 3.7. *For $v \in V(Q_d)$, a vertex u of weight m is within distance $m - 1$ of v if and only if $|u \cap v| \ge \frac{|v|+1}{2}$.*

Proof. The distance between any two vertices is their symmetric difference. Always the size of the symmetric difference is $|u| + |v| - 2\,|u \cap v|$. When $|u| = m$, it follows that $d(u, v) \le m - 1$ is equivalent to $|u \cap v| \ge \frac{|v|+1}{2}$. □

Lemma 3.8. *Let X be a subset of $[d]$ with $t = |X| \ge 2m$. For $u \in V(Q_d)$, let Y_u denote the set of vertices contained in X that have weight m and are within distance $m - 1$ of u in Q_d. If v is a vertex of Q_d with $|v| \ne 1$, then $|Y_v| \le \binom{t-3}{m-3} + 3\binom{t-3}{m-2}$.*

Proof. To facilitate comparisons of expressions involving binomial coefficients, note that $\binom{t-3}{m-3} + 3\binom{t-3}{m-2} = \binom{t-2}{m-2} + 2\binom{t-3}{m-2}$. Let $b_{t,m}$ denote this desired bound.

By Lemma 3.7, the vertices in Y_v are the m-sets contained in X that share at least $\frac{|v|+1}{2}$ elements with v. Thus $Y_v = \varnothing$ unles $|v \cap X| \ge \frac{|v|+1}{2}$, so we restrict our attention to that case. If $|v| = 1$ and $v \subseteq X$, then $|Y_v| = \binom{t-1}{m-1} > b_{t,m}$, which explains the restriction to $|v| \ne 1$. If $|v| \le 3$, then we require $|v \cap X| \ge 2$. If $|v \cap X| = 2$, then $|Y_v| = \binom{t-2}{m-2} < b_{t,m}$. If $|v| = 3$ with $v \subseteq X$, then $|Y_v| = \binom{t-3}{m-3} + 3\binom{t-3}{m-2}$.

For vertices of larger weight, we reduce the claim to studying vertices of odd weight. If $|v|$ is even and $i \in [d] - v$, then $|u \cap v| \ge \frac{|v|+1}{2}$ implies $|u \cap (v \cup \{i\})| \ge \frac{|v \cup \{i\}|+1}{2}$, since $|u \cap v|$ is an integer. Hence $Y_v \subseteq Y_{v \cup \{i\}}$, and it suffices to prove that $\left|Y_{v \cup \{i\}}\right| \le b_{t,m}$.

Hence we may assume that $|v|$ is odd. Having proved the bound when $|v| = 3$, we may assume that $|v|$ is a larger odd number, and it suffices to prove that $\left|Y_{v - \{i,j\}}\right| \ge |Y_v|$, where i

and j are distinct elements of v. If $|v| > 2m$, then $Y_v = \varnothing$, so we may assume $|v| < 2m \le t$. Let $v' = v - \{i, j\}$. It suffices to show $|Y_{v'} - Y_v| \ge |Y_v - Y_{v'}|$. We compute both sizes.

If $w \in Y_v - Y_{v'}$, then $|w \cap v| \ge \frac{|v|+1}{2}$ and $|w \cap v'| < \frac{|v'|+1}{2} = \frac{|v|-1}{2}$. This requires $\{i, j\} \subset w$ and $|w \cap v'| = \frac{|v|-3}{2}$, and hence $|w \cap v| = \frac{|v|+1}{2}$. Counting the ways to choose $w \cap v'$ and $w - v$, which together determine w, we have

$$|Y_v - Y_{v'}| = \binom{|v|-2}{\frac{|v|-3}{2}}\binom{|X-v|}{m-\frac{|v|+1}{2}}.$$

If $w \in Y_{v'} - Y_v$, then $|w \cap v'| \ge \frac{|v'|+1}{2} = \frac{|v|-1}{2}$ and $|w \cap v| < \frac{|v|+1}{2}$. This requires $|w \cap v| = |w \cap v'| = \frac{|v|-1}{2}$, so $i, j \notin w$. Again counting the ways to choose $w \cap v'$ and $w - v$, we have

$$|Y_{v'} - Y_v| = \binom{|v|-2}{\frac{|v|-1}{2}}\binom{|X-v|}{m-\frac{|v|-1}{2}}.$$

Since $\frac{|v|-3}{2} + \frac{|v|-1}{2} = |v| - 2$, the first factor is the same in both computations. Since $|v| < 2m \le t = |X|$, both second factors are positive. Also, $2m - (|v| - 1) \le |X - v| + 1$, with equality only if $v \subseteq X$ and $t = 2m$. In that case, the two second factors are the middle binomial coefficients and are equal; otherwise, $|Y_v - Y_{v'}| > |Y_{v'} - Y_v|$. $\square$

Theorem 3.9. *If $d \ge r \ge m \ge 3$ and $s \le r - \frac{3}{4}m^2$, then the revolutionaries win $\mathrm{RS}(Q_d, m, r, s)$, so $\sigma(Q_d, m, r) > r - \frac{3}{4}m^2$.*

Proof. Let the number of spies be $r - c$, where $c = 3m^2/4 \ge 2m$; we show that the revolutionaries win. The revolutionaries initially occupy the first r vertices of weight 1 in Q_d, threatening to make meetings after $m - 1$ moves at the vertices in $\binom{[r]}{m}$. A spy *watches* such a threat if its distance to the threat is at most $m - 1$, allowing it to arrive in time to cover the threatened meeting.

We speak only of the position at the end of the initial placement. It suffices to show that $r - c$ spies cannot watch all the threats at that time. Consider an initial placement of the $r - c$ spies that watches the maximum number of these threats. Let X be the set of indices of the vertices having uncovered revolutionaries. Let $t = |X|$; note that $t \ge c \ge 2m$. The $\binom{t}{m}$ threats made by these uncovered revolutionaries cannot be watched by spies on singleton vertices. We bound the number of such threats that can be watched by the $t - c$ spies on non-singleton vertices.

Let v be such a vertex. By Lemma 3.8, v watches at most $\binom{t-3}{m-3} + 3\binom{t-3}{m-2}$ of the threats from X. Hence these vertices watch at most $(t - c)[\binom{t-3}{m-3} + 3\binom{t-3}{m-2}]$ threats. Since the value of t can be set by the spies to anything between c and r, it suffices to show that for all such t this quantity is less than $\binom{t}{m}$.

With $\binom{t-3}{m-2} = \frac{t-m}{m-2}\binom{t-3}{m-3}$, the bound becomes $(t - c)\binom{t-3}{m-3}\left[1 + 3\frac{t-m}{m-2}\right]$. After using $\binom{t}{3}\binom{t-3}{m-3} = \binom{t}{m}\binom{m}{3}$ to substitute for $\binom{t-3}{m-3}$, we clear fractions and cancel $m - 2$ to turn the desired inequality into

$$m(m-1)(t-c)(3t-2m-2) < t(t-1)(t-2).$$

Since $m > 2$, it suffices to show $m(m-1)(t-c)(3t-6) < t(t-1)(t-2)$, which is equivalent to $3m(m-1)(t-c) < t(t-1)$. Writing this as $3m(t-c) < t\frac{t-1}{m-1}$ and noting that $\frac{t}{m} < \frac{t-1}{m-1}$ when $t > m$, the inequality $t \geq 2m$ implies that it suffices to have $3m(t-c) \leq t^2/m$. This is equivalent to $c \geq t(3m^2-t)/3m^2$, which holds for all t when $c = \frac{3}{4}m^2$. With this choice of c, we have proved that no value of t allows the spies to watch all the threats. □

We do not know whether the bound in Theorem 3.9 is anywhere near sharp. It seems that the revolutionaries win against more than $r - \frac{3}{4}m^2$ spies even with this simple strategy, because watching all the threats with this many spies on triples requires a combinatorial design that in general does not exist, just as in Theorem 3.5 we could not decompose K_4 into two triangles. Even when there are enough spies to watch all these threats, still a more sophisticated strategy may enable the revolutionaries to win. We think that $\sigma(Q_d, m, r) > r - \alpha m$ for some constant α when d is large.

4 Complete k-partite Graphs

In this section we obtain lower and upper bounds on $\sigma(G, m, r)$ when G is a complete k-partite graph. The lower bound requires partite sets large enough so that the revolutionaries can always access as many vertices in each part as they might want (enough to "swarm" to distinct vertices there that avoid all the spies). The upper bounds apply in more generality; they don't require large partite sets, and they require only a spanning k-partite subgraph (if there are additional edges within parts, then spies will be able to follow revolutionaries along them when needed).

Definition 4.1. A complete k-partite graph G is *r-large* if every part has at least $2r$ vertices. At the revolutionaries' turn on such a graph, an *i-swarm* is a move in which the revolutionaries make as many new meetings of size m as possible in part i. All revolutionaries outside part i move to part i, greedily filling uncovered partial meetings to size m and then making additional meetings of size m from the remaining incoming revolutionaries. When G is r-large, sufficient vertices are available in part i to permit this.

Theorem 4.2. *Let G be an r-large complete k-partite graph. If $k \geq m$, then $\sigma(G, m, r) \geq \frac{k}{k-1}\frac{k\lfloor r/k\rfloor}{m+c} - k$, where $c = 1/(k-1)$. When $k \mid r$ the bound simplifies to $\frac{k}{k-1}\frac{r}{m+c} - k$.*

Proof. We may assume that $k \mid r$, since otherwise the revolutionaries can play the strategy for the next lower multiple of k, ignoring the extra revolutionaries.

Let $t = r/k$. The revolutionaries initially occupy t distinct vertices in each part. Let s_i be the initial number of spies in part i. We may assume that they cover $\min\{s_i, t\}$ distinct revolutionaries, since each vertex of part i has the same neighborhood, and within part i these are the best locations. We compute the number of spies needed to avoid losing by swarm on round 1.

Case 1: $s_i > t$ *for some* i. If the revolutionaries swarm to part i, then all revolutionaries previously in part i are covered, so new meetings consist entirely of incoming revolutionaries and are not coverable by spies from part i. Since $(k-1)t$ revolutionaries arrive, at least $\lfloor (k-1)t/m \rfloor$ spies must arrive from other parts to cover the new meetings. Thus

$$s \geq s_i + \left\lfloor \frac{(k-1)t}{m} \right\rfloor \geq t\left(1 + \frac{k-1}{m}\right) = \frac{k-1+m}{k}\frac{r}{m}.$$

Case 2: $s_i \leq t$ *for all* i. For each i, part i has $t - s_i$ partial meetings. Since $s_i \geq 0$, an i-swarm is guaranteed to fill them if $(k-1)t \geq t(m-1)$, which holds when $k \geq m$. Hence the new meetings include all revolutionaries except the s_i covered by spies in part i before the swarm. Spies from other parts must cover $\lfloor (r - s_i)/m \rfloor$ new meetings in part i. Summing $s - s_i \geq (r - s_i - m + 1)/m$ over all parts yields $(k - 1 + 1/m)s \geq k(r - m + 1)/m$, so

$$s \geq \frac{k(r-m+1)}{m(k-1)+1} > \frac{k}{k-1}\frac{r}{m+c} - k.$$

The lower bound in Case 2 is smaller (better for spies) than the lower bound in Case 1, so the spies will prefer to play that way. The lower bound in Case 2 is thus a lower bound on $\sigma(G, m, r)$. □

The idea behind our strategies for spies is two-fold:
(1) find an invariant that prevents the revolutionaries from winning on the next round, and
(2) show that the spies can respond to the moves by revolutionaries to restore that invariant at the end of each round.

In general, we view a position satisfying the magic invariant(s) as being "stable", since it allows the spies to control the situation forever. Hence we re-use this term from Section 2.

Definition 4.3. Given a game position, say that m specified revolutionaries in a meeting and one spy covering them are *bound*. After specifying the bound players for all vertices hosting meetings, the remaining spies and revolutionaries are *free* (we specify $\lfloor t/m \rfloor$ meetings at a vertex having t revolutionaries).

Let $\hat{r}_i$ and $\hat{s}_i$ denote the numbers of free revolutionaries and free spies in part i in the current position of a game on a complete k-partite graph. Let $\hat{r}$ and $\hat{s}$ denote the total numbers of free revolutionaries and free spies. A game position is *stable* if (1) all meetings are covered, and (2) $\hat{s} - \hat{s}_i \geq \hat{r}/m$ for each part i.

Lemma 4.4. *Let G be a graph having a spanning complete k-partite subgraph G'. If the position at the end of the previous round was stable for G', then the revolutionaries cannot win in the current round on G.*

Proof. After the revolutionaries move in the current round, some meetings (including old meetings) exist. We prove that these meetings can be matched to spies on vertices equal or adjacent to them. The spies can then move to cover all the meetings.

Create an auxiliary bipartite graph H in which partite set X is the set of vertices hosting meetings and partite set Y is the set of spies. For $x \in X$ and $y \in Y$, let xy be an edge in H if the spy y is at a vertex of $N_G[x]$. If H has a matching that covers X, then the spies can move (or remain in place) to cover all meetings at the end of the current round. Here the designation of a spies or revolutionary as free or bound indicates its status at the end of the previous round.

It suffices to verify Hall's Condition for a matching that covers X. For $S \subseteq X$, we show that $|N_H(S)| \geq |S|$. Note that $|X| \leq \lfloor r/m \rfloor \leq s$. If S has vertices from more than one partite set in the spanning complete k-partite subgraph G', then $|N_H(S)| = s \geq |X|$.

If S has vertices only from part i, then let t be the number of vertices in $N_{G'}[S]$ that hosted meetings before the revolutionaries moved. By stability, these vertices have bound spies, which lie in $N_H(S)$; furthermore, $\hat{s} - \hat{s}_i \geq \hat{r}/m$, and all of the free spies counted by $\hat{s} - \hat{s}_i$ are also in $N_H(S)$. No spy is both free and bound, so $|N_H(S)| \geq t + \hat{r}/m$. On the other hand, $|S| \leq \frac{\hat{r}+tm}{m}$, since the numerator is an upper bound on the number of revolutionaries that can be used to make meetings in S. Thus $|S| \leq |N_H(S)|$, as desired. □

Theorem 4.5. *If a graph G has a spanning complete k-partite subgraph, then $\sigma(G, m, r) \leq \left\lceil \frac{k}{k-1} \frac{r}{m} \right\rceil + k$.*

Proof. Let G' be the specified subgraph, and let $s = \left\lceil \frac{k}{k-1} \frac{r}{m} \right\rceil + k$. It suffices to show that s spies can produce a stable position at the end of each round. First, after the revolutionaries have moved, the spies cover all newly created meetings, moving the fewest possible spies to do so. By Lemma 4.4, the spies can do this since the previous round ended in a stable position (also, $s \geq \lfloor r/m \rfloor$ guarantees that the spies can do this in the initial position).

Next, the spies that are now free distribute themselves equally among the k parts of G'. More precisely, with $\hat{s}$ being the total number of free spies after the new meetings are covered and $\hat{s}_i$ being the number of them in part i, we have $|\hat{s}_i - \hat{s}/m| < 1$ for all i.

It suffices to show that this second step produces a stable position. In order to have $\hat{s} - \hat{s}_i \geq \hat{r}/m$ for all i, it suffices to have $\hat{s}_j \geq \hat{r}/[m(k-1)]$ for each j. Since the free spies are distributed equally, it suffices for the average to be big enough: $\hat{s}/k \geq \hat{r}/[m(k-1)] + 1$. Multiplying by k, we require $\hat{s} \geq \frac{k}{k-1} \frac{\hat{r}}{m} + k$.

We are given $s \geq \frac{k}{k-1} \frac{r}{m} + k$. The number of bound revolutionaries is exactly m times the number of bound spies; hence $s - \hat{s} = (r - \hat{r})/m$. Subtracting this equality from the given inequality yields

$$\hat{s} \geq \frac{1}{k-1}\frac{r}{m} + \frac{\hat{r}}{m} + k \geq \frac{k}{k-1}\frac{\hat{r}}{m} + k,$$

where the last inequality uses $r \geq \hat{r}$. We now have the inequality that we showed suffices for a stable position. □

5 Complete Bipartite Graphs

Finally, let G be an r-large bipartite graph. We give lower and upper bounds on $\sigma(G, m, r)$ for fixed m. The lower bound strategies for the revolutionaries win after one or two rounds, while the upper bounds use more delicate strategies for the spies (maintaining invariants that prevent the revolutionaries from winning on the next round).

Since the lower bounds are much easier, we start with them, but first we compare all the bounds in Table 1. When $3 \mid m$, the lower bound is roughly $\frac{3}{2}r/m$. We believe that this is the asymptotic answer when $3 \mid m$. When $3 \nmid m$, the revolutionaries cannot employ this strategy quite so efficiently, which leaves an opening for the spies to do better. Indeed, for $m = 2$, the answer is roughly $\frac{7}{5}r/m$, a bit smaller. For larger m, the relative value of this advantage diminishes, and we expect the leading coefficient to tend to 3/2 as $m \to \infty$.

Table 1: Bounds on $\sigma(G, m, r)$

Meeting size	Lower bound	Upper bound	References
2	$\left\lceil \frac{\lfloor 7r/2 \rfloor - 3}{5} \right\rceil$	$\left\lceil \frac{\lfloor 7r/2 \rfloor - 3}{5} \right\rceil$	Theorems 5.2 and 5.9
3	$\lfloor r/2 \rfloor$	$\lfloor r/2 \rfloor$	Theorems 5.3 and 5.10
$m \in \{4, 8, 10\}$	$\frac{1}{5}\lfloor 7r/m \rfloor - 2$		Corollary 5.4
m	$\left\lfloor \frac{1}{2} \left\lfloor \frac{r}{\lceil m/3 \rceil} \right\rfloor \right\rfloor$	$\left(1 + \frac{1}{\sqrt{3}}\right)\frac{r}{m} + 1$	Corollary 5.4; Theorem 5.11

We first motivate the lower bounds by giving simple strategies for the revolutionaries when $m \in \{2, 3\}$. Henceforth call the partite sets X_1 and X_2.

Example 5.1. Initially place $\lfloor r/2 \rfloor$ revolutionaries in X_1 and $\lceil r/2 \rceil$ revolutionaries in X_2. Regardless of where the spies sit, swarming revolutionaries can form at least $\lfloor (r-1)/(2m) \rfloor$ new meetings on either side that can only be covered by spies from the other side, so the initial placement must satisfy $s_1 \geq \lfloor (r-1)/(2m) \rfloor$ and $s_2 \geq \lfloor r/(2m) \rfloor$, where s_i is the number of spies in X_i.

However, the uncovered revolutionaries can also be used to form meetings. If $m = 2$, then the revolutionaries can form $\lfloor (r - s_i)/2 \rfloor$ meetings when swarming to X_i, so the spies lose unless $s_{3-i} \geq \lfloor (r - s_i)/2 \rfloor$ for both i. Summing the inequalities yields $s_1 + s_2 \geq 2(r-1)/3$.

For $m = 3$, considering only r of the form $4k$, where $k \in \mathbb{N}$, we show that the revolutionaries win against $2k-1$ spies. Initially there are $2k$ revolutionaries in each part, on distinct vertices. We may assume $s_1 \le s_2$, so $s_1 \le k-1$. Since there are only $2k-1-s_1$ spies in X_2, there are at least s_1+1 uncovered revolutionaries in X_2. Since $s_1 \le k-1$, we can use $2(s_1+1)$ revolutionaries from X_1 to form meetings of size 3 with the uncovered revolutionaries in X_2. Since only s_1 spies are available to cover these meetings, the spies lose.

Thus $\sigma(G, 3, r) \ge r/2$ when $4 \mid r$. However, when $r = 4k+2$, the revolutionaries cannot immediately win against $2k$ spies by this construction. With $2k+1$ revolutionaries in each part and k spies sitting on revolutionaries in each part, swarming revolutionaries can only make k new meetings in either part, which can be covered by the spies. □

The symmetric strategy in Example 5.1 is optimal when $m = 3$ and $4 \mid r$. However, when $m = 2$ and when $m = 3$ with $r = 4k+2$, the revolutionaries can do better using an asymmetric strategy that takes advantage of moving away from spies. When $m = 3$ and $r = 4k+2$, this other strategy just increases the threshold by 1, to the value $\lfloor r/2 \rfloor$ that we will show is optimal for all r. For $m = 2$, however, the better strategy increases the leading term from $2r/3$ to $7r/10$.

Recall that the partite sets are X_1 and X_2 and that a vertex (or meeting) is *covered* if there is a spy there. Say that a spy is *lonely* when at a vertex with no revolutionary.

Theorem 5.2. *If G is an r-large complete bipartite graph, then $\sigma(G, 2, r) \ge \left\lceil \frac{\lfloor 7r/2 \rfloor - 3}{5} \right\rceil$.*

Proof. We present a strategy for the revolutionaries and compute the number of spies needed to resist it. The revolutionaries start at r distinct vertices in X_1. In response, at least $\lfloor r/2 \rfloor$ spies must start in X_1, since otherwise the revolutionaries can next make $\lfloor r/2 \rfloor$ meetings at uncovered vertices in X_2 and win.

In the first round, $\lfloor r/2 \rfloor$ revolutionaries move from X_1 to X_2, occupying distinct vertices. They leave from vertices of X_1 that are covered by spies (as much as possible), so after they move at least $\lfloor r/2 \rfloor$ spies in X_1 are lonely. Now the spies move; let s_i be the number of spies in X_i after they move (for $i \in \{1, 2\}$). Let c be the number of revolutionaries in X_1 that are now covered by spies. Since at most s_2 spies leave X_1, there remain at least $\lfloor r/2 \rfloor - s_2$ lonely spies in X_1. We conclude that $c \le s_1 - \lfloor r/2 \rfloor + s_2$.

In round 2, the revolutionaries have the opportunity to swarm to X_1 or X_2. Since there are $\lfloor r/2 \rfloor$ revolutionaries in X_2, there are at most $\lfloor r/2 \rfloor + 1$ uncovered revolutionaries in X_1 (on distinct vertices), so swarming revolutionaries can make meetings with all but at most 1 uncovered revolutionary in X_1. The revolutionaries can therefore make $\lfloor (r-c)/2 \rfloor$ new meetings in X_1. These meetings can only be covered by spies moving from X_2, so the spies lose unless $s_2 \ge \lfloor (r-c)/2 \rfloor$.

If the revolutionaries swarm to X_2, then the new meetings there can only be covered by spies coming from X_1. At most s_2 revolutionaries in X_2 are covered by spies. Since $\lceil r/2 \rceil$

revolutionaries come from X_1, they can make meetings with all uncovered revolutionaries in X_2, so the spies lose unless $s_1 \geq \lfloor (r - s_2)/2 \rfloor$.

Adding twice the lower bound on s_1 to the lower bound on s_2 (with $c \leq s_1 - \lfloor r/2 \rfloor + s_2$),

$$s_2 + 2s_1 \geq \frac{\lfloor 3r/2 \rfloor - s_1 - s_2 - 1}{2} + r - s_2 - 1.$$

The inequality simplifies to $5(s_1 + s_2) \geq \lfloor 7r/2 \rfloor - 3$, as desired. □

The general lower bound in Corollary 5.4 uses the formula for $m = 3$, which we study first. The key is that $r/2 - 1$ spies are not enough when $r \equiv 2 \mod 4$; we first sketch the idea in an easy case. Suppose that $r = 4k + 2 \equiv 6 \mod 12$. The revolutionaries start at distinct vertices in X_1. Suppose that all s spies start in X_1 and that there are enough of them to win. In round 1, $2r/3$ revolutionaries move to X_2, leaving the spies in X_1 lonely. Let s_2 be the number of spies that move to X_2 after round 1, leaving s_1 spies in X_1. The revolutionaries in X_2 now can make $r/3$ meetings with the remaining $r/3$ revolutionaries in X_1, so $s_2 \geq r/3$. Since $s_2 \leq 2k = r/2 - 1$, at least $r/6 + 1$ revolutionaries remain uncovered in X_2. The remaining $r/3$ revolutionaries in X_1 can make meetings with $r/6$ of them in round 2. Hence $s_1 \geq r/6$, and $s = s_1 + s_2 \geq r/2$.

The initial placement only requires $r/3$ spies in X_1, not $r/2$. We must allow for initial placement of x spies in X_2, where $0 \leq x \leq r/6$. The x spies originally in X_2 can move to X_1 in round 1 and cover revolutionaries there; this prevents the revolutionaries from threatening as many meetings by a swarm to X_1. In response, fewer than $2r/3$ revolutionaries move to X_2 in round 1, and yet we can guarantee more threatened meetings in the swarm to X_2.

Theorem 5.3. *If G is an r-large complete bipartite graph, then $\sigma(G, 3, r) \geq \lfloor r/2 \rfloor$.*

Proof. Since $\lfloor r/2 \rfloor = \lfloor (r+1)/2 \rfloor$ when r is even, and having an extra revolutionary cannot reduce σ, it suffices to prove the lower bound when r is even. Example 5.1 proves it when $4 \mid r$, so only the case $r = 4k + 2$ remains. We show that $4k + 2$ revolutionaries can win against $2k$ spies.

The revolutionaries start at r distinct vertices of X_1, so at least $\lfloor r/3 \rfloor$ spies must start in X_1. Let x be the initial number of spies in X_2, with $2k - x$ spies in X_1. Since X_1 contains at least $\lfloor r/3 \rfloor$ spies, $x \leq \lceil (2k-2)/3 \rceil = \lceil r/6 \rceil - 1$. Define j by $r - x \equiv j \mod 3$ with $j \in \{0, 1, 2\}$. In round 1, p revolutionaries move to X_2, where $p = 2(r - x - j)/3$. Note that $p \geq 2k - x$, so all spies in X_1 are now lonely. The number of revolutionaries remaining in X_1 is $r - p$, which equals $(r + 2x + 2j)/3$.

Let s_i be the number of spies in X_i after the spies respond in round 1. Since at most x spies move from X_2 to X_1 in round 1, the number of uncovered revolutionaries in X_1 is now at least $(r - x + 2j)/3$. With $p = 2(r - x - j)/3$, there are enough revolutionaries in X_2 to threaten meetings with $(r - x - j)/3$ vertices in X_1. Hence $s_2 \geq (r - x - j)/3$.

Now consider a swarm to X_2. Since there were $2k - x$ spies in X_1 initially, the number who moved to X_2 and covered revolutionaries after round 1 is at most $2k - x$. Hence X_2 has at least $p - 2k + x$ uncovered revolutionaries. The $r - p$ revolutionaries remaining in X_1 can generate enough pairs to make meetings with this many uncovered revolutionaries in X_2 when $j \neq 0$. We have $r - p = (r + 2x + 2j)/3$ and $p - 2k + x = (r + 2x + 6 - 4j)/6$.

If $j = 0$, then we can only make $p - 2k + x - 1$ meetings in the swarm to X_2, so $s_1 \geq (r+2x)/6$, and we obtain $s_2+s_1 \geq \frac{r-x}{3}+\frac{r+2x}{6} = r/2$. If $j = 1$, then we can make $p-2k+x$ meetings in the swarm, so $s_1 \geq (r+2x+2)/6$, and we obtain $s_2+s_1 \geq \frac{r-x-1}{3}+\frac{r+2x+2}{6} = r/2$. Finally, if $j = 2$, then the same computation yields only $s \geq \frac{r-x-2}{3} + \frac{r+2x-2}{6} = r/2 - 1$. However, equality holds only if all $2k - x$ spies initially in X_1 move to X_2 in round 1 to cover revolutionaries. Only x spies remain in X_1 to guard the swarm to X_2 that makes $(r + 2x - 2)/6$ meetings. The inequality $x \geq (r + 2x - 2)/6$ requires $x \geq (r - 2)/4$, but guarding the initial position required $x < r/6$. □

Corollary 5.4. *If G is an r-large complete bipartite graph, then $\sigma(G, m, r) \geq \left\lfloor \frac{1}{2} \left\lfloor \frac{r}{\lceil m/3 \rceil} \right\rfloor \right\rfloor$. If m is even, then $\sigma(G, m, r) \geq \frac{1}{5} \lfloor 7r/m \rfloor - 2$.*

Proof. Let $m' = \lceil m/3 \rceil$. The revolutionaries group into cells of size m'; each cell moves together, modeling one player in a game with meeting size 3. When three of these cells converge to make an unguarded meeting, the revolutionaries win the original game. The r revolutionaries make $\lfloor r/m' \rfloor$ such cells and ignore extra revolutionaries. By Theorem 5.3, the number of spies needed to keep the revolutionaries from winning is at least $\lfloor \lfloor r/m' \rfloor /2 \rfloor$.

For even m, let $m' = m/2$. The revolutionaries can group into $\lfloor r/m' \rfloor$ cells of size m' and play a game with meeting size 2. By Theorem 5.2, the lower bound is now $\frac{1}{5}(\lfloor (7 \lfloor 2r/m \rfloor /2 - 3 \rfloor)$. This improves on the bound above when $m \in \{4, 8, 10\}$. □

Finally, we consider upper bounds for $\sigma(G, m, r)$ when G is an r-large bipartite graph, proved by giving strategies for the spies.

Definition 5.5. Henceforth, always G is a r-large bipartite graph with partite sets X_1 and X_2, and we consider the game $\mathrm{RS}(G, m, r, s)$. Any statement that includes index j is considered for both $j = 1$ and $j = 2$. The numbers of revolutionaries and spies in part j at the beginning of the current round are denoted by r_j and s_j, respectively, and the number of revolutionaries in part j that are on vertices covered by spies is denoted by c_j. The corresponding counts at the end of the round are denoted by r'_j, s'_j and c'_j.

A spy that moves to X_j during the round is *new*; spies that remained in X_j and did not move are *old*. A meeting formed at a vertex in X_j during the round is *new* if at the end of the previous round there was no meeting there; a meeting is *old* if it is not new. The revolutionaries *swarm* X_j in a round if at the end of the round all revolutionaries are in X_j.

Definition 5.6. A *greedy migration strategy* is a strategy for the spies having the following properties. First, no vertex ever has more than one spy on it. Next, after the revolutionaries move during the current round and the spies compute the new desired distribution s_1', s_2' of spies on X_1 and X_2, they move to reach that distribution as follows. Since $s_1' + s_2' = s_1 + s_2$, by symmetry there is an index $i \in \{1, 2\}$ such that $s_i' \leq s_{3-i}$. The spies reach their locations for the end of the round via the following steps.

(1) s_i' spies move away from X_{3-i}, iteratively leaving vertices that now have the fewest revolutionaries among those in X_{3-i}.

(2) *All* s_i spies previously on X_i leave X_i and move to uncovered vertices in X_{3-i}, iteratively covering vertices having the most revolutionaries.

(3) The s_i' spies that left X_{3-i} now move to uncovered vertices in X_i, iteratively covering vertices having the most revolutionaries.

Remark 5.7. At the end of each round under a greedy migration strategy, for each j either all spies in X_j are new, or all spies that were in X_{3-j} have migrated to part j. In the specification of the movements in Definition 5.6, the former occurs when $j = i$, and the latter occurs when $j = 3 - i$. In the first case, there are s_j' new spies in X_j; in the second case, there are s_{3-j} new spies in X_j. In particular, after each round at least $\min\{s_j', s_{3-j}\}$ spies in X_j are new.

Lemma 5.8. *Any greedy migration strategy in* $\mathrm{RS}(G, m, r, s)$ *is a winning strategy for the spies if it prevents the revolutionaries from winning by swarming a part.*

Proof. Consider the end of a round, with r_j and s_j counting the revolutionaries and spies in X_j. We show that if the revolutionaries do not win by swarming on the next round, then all meetings in X_j are now covered. Thus the hypothesis implies that at the end of every round all meetings are covered.

If the r_j revolutionaries on part j swarm to X_{3-j}, then they can form at least $\lfloor r_j/m \rfloor$ new meetings at uncovered vertices. Such meetings can only be covered by new spies coming from X_j, so the hypothesis requires $s_j \geq \lfloor r_j/m \rfloor$. Since greedy migration places spies on r_j to maximize coverage, if there are $\lfloor r_j/m \rfloor$ new spies they cover all meetings. Hence we may assume that not all spies in X_j are new.

Now Remark 5.7 implies that all the spies that were previously in X_{3-j} migrated to X_j in this round. We claim that the number s_{3-j} of those spies is at least the number of new meetings in X_j. Otherwise, the revolutionaries have now won by making a number of new meetings that is at most what they can make by swarming X_j on this round, and the s_{3-j} spies could not defend against that swarm. Hence it suffices to show that the old meetings in X_j continue to be covered by old spies.

Suppose that a spy leaves an old meeting in X_j. Since greedy migration picks departing spies to minimize the number of revolutionaries uncovered, all old spies that remain in part

j are covering meetings. All the new spies are placed in X_j to maximize coverage, so if there is an uncovered meeting in X_j at the end of this round, then every spy in X_j is covering a meeting. This now contradicts the value of r_j, since $s_j \ge \lfloor r_j/m \rfloor$. □

Theorem 5.9. *If G is an r-large complete bipartite graph, then $\sigma(G,2,r) \le \lceil \frac{\lfloor 7r/2 \rfloor - 3}{5} \rceil$.*

Proof. Let $s = \lceil \frac{\lfloor 7r/2 \rfloor - 3}{5} \rceil$; we give a winning strategy for the spies in RS$(G,2,r,s)$. Let $\alpha = s - \lfloor r/2 \rfloor$ and $\beta = \lfloor (r-\alpha)/2 \rfloor$. Later we will use the following inequalities: $\alpha \le \beta$, $\alpha + \beta \le s$, and $\lfloor (r+\beta)/2 \rfloor \le s$. These inequalities can be checked explicitly for each congruence class modulo 10. The first two are loose, since $\alpha \approx 2r/10$, $\beta \approx 4r/10$, and $s \approx 7r/10$, but the third is delicate, with equality holding in except in two congruence classes and the floor function needed for correctness in four congruence classes.

During the game, if the revolutionaries swarm X_{3-j} in the current round, then they generate at most $\min\{r_j, \lfloor \frac{r-c_{3-j}}{2} \rfloor\}$ new meetings. The spy strategy will ensure

$$s_j \ge \min\left\{r_j, \left\lfloor \frac{r - c_{3-j}}{2} \right\rfloor\right\} \qquad \text{for } j \in \{1,2\}, \tag{A}$$

and hence it will keep the revolutionaries from winning by a swarm. The spies move by greedy migration after computing the new values s_1' and s_2' in response to r_1' and r_2'. By Lemma 5.8, the spies win by a greedy migration strategy that keeps the revolutionaries from winning by swarm.

The spies determine s_1' and s_2' via three cases, using the first that applies. Always $s_1' + s_2' = s$.

Case 1: If $r_i' \le \alpha$ for some $i \in \{1,2\}$, then $s_i' = \alpha$.

Case 2: If $s_i \ge \min\{r_{3-i}', \beta\}$ for some $i \in \{1,2\}$, then $s_{3-i}' = \min\{r_{3-i}', \beta\}$.

Case 3: Otherwise, $s_i' = s_{3-i}$ and $s_{3-i}' = s_i$.

It remains to prove (A). In order to do so, we first prove

$$s_j \ge \alpha \qquad \text{for } j \in \{1,2\}. \tag{B}$$

Trivially the spies can satisfy both (A) and (B) in round 0. Assuming that these invariants hold before the current round begins, we will show that they also hold when it ends.

Invariant (B) is preserved. In Case 1, $s_i' = \alpha$ and $s_{3-i}' = \lfloor r/2 \rfloor > \alpha$. In Case 3, $s_j' = s_{3-j} \ge \alpha$. In Case 2, $r_{3-i}' > \alpha$, so $s_{3-i}' = \min\{r_{3-i}', \beta\} \ge \alpha$, and $s_i' = s - s_{3-i}' = s - \min\{r_{3-i}', \beta\} \ge s - \beta \ge \alpha$.

Invariant (A) is preserved. In Case 1, $s_i' = \alpha \ge r_i' \ge \min\{r_i', \left\lfloor \frac{r - c_{3-i}'}{2} \right\rfloor\}$ and $s_{3-i}' = \lfloor r/2 \rfloor \ge \left\lfloor \frac{r - c_{3-i}'}{2} \right\rfloor \ge \min\{r_i', \left\lfloor \frac{r - c_{3-i}'}{2} \right\rfloor\}$.

In Case 2 with $s_i \geq \min\{r'_{3-i}, \beta\}$, first consider $j = 3 - i$. We have $s'_{3-i} = \min\{r'_{3-i}, \beta\}$. If $s'_{3-i} = r'_{3-i}$, then s'_{3-i} is already big enough, so suppose $s'_{3-i} = \beta$. By Remark 5.7, at least $\min\{s'_i, s_{3-i}\}$ spies in X_i are new. By (B), this quantity is at least α, and Case 2 requires $r'_i > \alpha$. Hence the new spies cover at least α revolutionaries, and $c'_i \geq \alpha$ yields $s'_{3-i} = \beta = \left\lfloor \frac{r-\alpha}{2} \right\rfloor \geq \min\{r'_{3-i}, \left\lfloor \frac{r-c'_i}{2} \right\rfloor\}$.

Now consider $j = i$. By Remark 5.7, at least $\min\{s_i, s'_{3-i}\}$ spies in X_i are new, and in Case 2 each of s_i and s'_{3-i} is at least $\min\{r'_{3-i}, \beta\}$. Since spies cover greedily, $c'_{3-i} \geq \min\{r'_{3-i}, \beta\} = s'_{3-i}$. Also $s'_{3-i} \leq \beta$, so

$$s'_i = s - s'_{3-i} \geq \left\lfloor \frac{r+\beta}{2} \right\rfloor - s'_{3-i} \geq \left\lfloor \frac{r - s'_{3-i}}{2} \right\rfloor \geq \left\lfloor \frac{r - c'_{3-i}}{2} \right\rfloor \geq \min\left\{ r'_i, \left\lfloor \frac{r - c'_{3-i}}{2} \right\rfloor \right\}. \quad (8)$$

Finally, $s'_j = s_{3-j} < \min\{r'_j, \beta\}$ in Case 3, since Case 2 does not apply. Since all spies move and $s'_j \leq r'_j$, we have $c'_j \geq s'_j$. Hence for each j the computation in (8) is valid. □

The method for the upper bound when $m = 3$ is essentially the same.

Theorem 5.10. *If G is an r-large complete bipartite graph, then $\sigma(G, 3, r) \leq \lfloor r/2 \rfloor$.*

Proof. We present a greedy migration strategy for $\lfloor r/2 \rfloor$ spies that keeps the revolutionaries from winning by swarming; by Lemma 5.8 it is a winning strategy for the spies.

Define r_j, s_j, c_j at the start of a round and r'_j, s'_j, c'_j at the end of the round in the same way as before. Also, we need to know the maximum number of revolutionaries together on an uncovered vertex in X_j at the beginning and end of the round; let these values be u_j and u'_j. If the revolutionaries have not already won, then $u_j, u'_j \leq 2$. Let $s = \lfloor r/2 \rfloor$, $\alpha = \lfloor r/2 \rfloor - \lfloor r/3 \rfloor$, and $\beta = s - \lfloor (r - \alpha)/3 \rfloor$. We will want the inequalities $\beta \geq \left\lfloor \frac{r-2\alpha}{3} \right\rfloor$ and $\beta \leq \left\lceil \frac{\lfloor r/2 \rfloor}{2} \right\rceil$. The latter is always satisfied (the left side is about $2r/9$ and the right side is about $r/4$), but both sides of the first inequality are about $2r/9$. Checking each congruence class modulo 18 shows that $\beta \geq \left\lfloor \frac{r-2\alpha}{3} \right\rfloor$ except when $r \equiv 3 \mod 18$.

The values s'_1 and s'_2 that determine the movements of spies in this round under the greedy migration strategy are computed as follows, with $s'_{3-i} = s - s'_i$ always. Note that since $r'_1 + r'_2 = r$, one case below holds for exactly one index, except when $r'_1 = r'_2 = r/2$, in which case it does not matter which we call i.

Case 1: If $r'_i \leq \alpha$ for some $i \in \{1, 2\}$, then $s'_i = \alpha$.

Case 2: If $\alpha < r'_i \leq \beta$ for some $i \in \{1, 2\}$, then $s'_i = r'_i$.

Case 3: If $\beta < r'_i \leq 2\beta$ for some $i \in \{1, 2\}$, then $s'_i = \beta$, except $s'_i = \beta + 1$ when $s_i = \alpha$.

Case 4: If $2\beta < r'_i \leq \lfloor r/2 \rfloor$ for some $i \in \{1, 2\}$, then $s'_i = \lfloor r'_i/2 \rfloor$.

Let $f_j = \min\{\lfloor\frac{r-c_{3-j}}{3}\rfloor, \lfloor\frac{r_j}{3-u_{3-j}}\rfloor\}$. During the game, if the revolutionaries swarm X_{3-j} in the current round, then they generate at most f_j new meetings. Hence it suffices to show that the strategy specified above always ensures

$$s_j \geq f_j \qquad \text{for } j \in \{1,2\}. \tag{A}$$

As in Theorem 5.9, in order to prove (A) we will also need

$$s_j \geq \alpha \qquad \text{for } j \in \{1,2\}. \tag{B}$$

Place the spies to satisfy (A) and (B) in round 0. In each Case of play, $\alpha \leq s'_i \leq \lfloor r/4 \rfloor \leq s-\alpha$, so (B) is preserved. Now $s_1, s_2, s'_1, s'_2 \geq \alpha$, and we study (A).

With f'_j being the value of f_j at the end of the round, we need $s'_j \geq f'_j$. By Remark 5.7, each part receives at least α new spies in each round. In Cases 2, 3, and 4 each part contains at least α revolutionaries, so $c'_j \geq \alpha$ in those cases. Also $s'_j \geq \lfloor r'_j/3 \rfloor$ in each Case. Since $s'_j \geq \lfloor r'_j/3 \rfloor = \lfloor r'_j/(3-u'_{3-j}) \rfloor$ when $u'_{3-j} = 0$, we may assume $u'_j \in \{1,2\}$.

In addition, since the greedy strategy places new spies in X_j to maximize coverage, leaving an uncovered vertex with u'_j revolutionaries implies that each of the (at least) α new spies covers at least u'_j revolutionaries at its vertex. Hence $c'_j \geq u'_j\alpha$.

Invariant (A) is preserved:

In Case 1, $s'_i = \alpha \geq r'_i \geq f'_i$ and $s'_{3-i} = s - \alpha \geq \lfloor r/3 \rfloor \geq f'_{3-i}$.

In Case 2, $s'_i = r'_i \geq f'_i$. Also, $c'_i \geq \alpha$ and $s'_{3-i} = s - r'_i \geq s - \beta = \lfloor\frac{r-\alpha}{3}\rfloor \geq \lfloor\frac{r-c'_i}{3}\rfloor \geq f'_{3-i}$.

In Case 3, again $c'_i \geq \alpha$, so $s'_{3-i} = s - \beta = \lfloor\frac{r-\alpha}{3}\rfloor \geq \lfloor\frac{r-c'_i}{3}\rfloor \geq f'_{3-i}$.

In Case 3 or Case 4, if $u'_{3-i} = 1$, then $s'_i \geq \lfloor r'_i/2 \rfloor = \lfloor\frac{r'_i}{3-u'_{3-i}}\rfloor \geq f'_i$. If $u'_{3-i} = 2$, then $c'_{3-i} \geq 2\alpha$. Hence $s'_i \geq \beta \geq \lfloor\frac{r-2\alpha}{3}\rfloor \geq \lfloor\frac{r-c'_{3-i}}{3}\rfloor \geq f'_i$, with the exception that $\beta = \lfloor\frac{r-2\alpha}{3}\rfloor - 1$ when $r \equiv 3 \mod 18$. Either $s'_i > \beta$, which suffices, or $s_i > \alpha$. In the latter case, there are more than α new spies on X_{3-j}, so $c'_{3-i} \geq 2\alpha + 2$, which is enough to fix the problem since we are only worried when $r \equiv 3 \mod 18$.

In Case 4, if $u'_i = 1$, then $s'_{3-i} = s - \lfloor\frac{r'_i}{2}\rfloor \geq \lfloor\frac{r'_{3-i}}{2}\rfloor = \lfloor\frac{r'_{3-i}}{3-u'_i}\rfloor \geq f'_{3-i}$. If $u'_i = 2$, then $c'_i \geq 2\alpha$. Now $s'_{3-i} = s - \lfloor\frac{r'_i}{2}\rfloor \geq \lfloor\frac{r}{2}\rfloor - \lfloor\frac{\lfloor r/2 \rfloor}{2}\rfloor = \lceil\frac{\lfloor r/2 \rfloor}{2}\rceil \geq \lfloor\frac{r-2\alpha}{3}\rfloor \geq \lfloor\frac{r-c'_i}{3}\rfloor \geq f'_{3-i}$. □

Theorem 5.11. *If G is an r-large complete bipartite graph, and $\frac{r}{m} \geq \frac{1}{1-1/\sqrt{3}}$, then $\sigma(G,m,r) \leq (1 + \frac{1}{\sqrt{3}})\frac{r}{m} + 1$.*

Proof. For $s \geq (1 + \frac{1}{\sqrt{3}})\frac{r}{m} + 1$, we present a greedy migration strategy for s spies that keeps the revolutionaries from winning by swarming. As usual, r_j and s_j count the revolutionaries and spies in X_j to begin a round, r'_j counts the revolutionaries after they move, and s'_j is the number of spies to be computed for X_j to end the round. To determine s'_1 and s'_2, the spies compute x, α, u_1, and u_2 (not necessarily integers) such that

$$x \le \lfloor r/m \rfloor, \qquad x + r/m + 1 \le s, \qquad \text{and} \tag{9}$$

$$\alpha = x + r/m - \frac{r - u_1 x}{m} = x + r/m - \frac{r_2'}{m - u_1} = \frac{r_1'}{m - u_2} = \frac{r - u_2 x}{m}. \tag{10}$$

We will show that such numbers always exist. Now s_1' and s_2' are computed as follows:

Case 1: If $\alpha \le x$, then $s_1' = \lceil x \rceil$ and $s_2' = s - s_1'$.

Case 2: If $\alpha > \lfloor r/m \rfloor$, then $s_1' = \lfloor r/m \rfloor$ and $s_2' = s - s_1'$.

Case 3: If $x < \alpha \le \lfloor r/m \rfloor$, then $s_1' = \lceil \alpha \rceil$ and $s_2' = s - s_1'$.

Since always $s_j' \ge x$, greedy migration moves at least $\lceil x \rceil$ new spies to each part in each round, by Remark 5.7. Consider a swarm. If all uncovered vertices in X_j have at most u_j revolutionaries, then swarming X_j generates at most $r_{3-j}'/(m - u_j)$ new meetings. If some uncovered vertex in X_j has more than u_j revolutionaries, then by greedy migration at least x spies in X_j have covered more than u_j revolutionaries each, and swarming X_j forms at most $(r - u_j x)/m$ new meetings. Hence swarming X_j fails to win if

$$s_{3-j}' \ge \max\left\{ \frac{r_{3-j}'}{m - u_j}, \frac{r - u_j x}{m} \right\}. \tag{11}$$

For $j = 2$, both quantities on the right in (11) equal α, so the condition is equivalent to $s_1' \ge \alpha$, which holds in Cases 1 and 3. In Case 2, $s_1' = \lfloor r/m \rfloor$, which always protects against swarming X_2 since at most $\lfloor r/m \rfloor$ meetings can be made.

For $j = 1$, both quantities on the right in (11) equal $x + r/m - \alpha$, so the condition is equivalent to $s_2' \ge x + r/m - \alpha$. Since $s - 1 \ge x + r/m$, proving $s_2' \ge s - 1 - \alpha$ shows that swarming X_1 is ineffective. In Case 1, $s_2' > r/m$, which suffices. In Case 2 or 3, $s_1' \le \lceil \alpha \rceil$, so $s_2' \ge s - \lceil \alpha \rceil > s - 1 - \alpha$, as desired.

It remains to show that such numbers exist. Solving (10) yields

$$x = \frac{\sqrt{9r^2 + 12r_1'r - 12{r_1'}^2}}{6m}$$

$$u_1 = \frac{r + mx - \sqrt{r^2 + 2rxm + x^2m^2 - 4xr_1'm}}{2x} \quad \text{and}$$

$$u_2 = \frac{r + mx - \sqrt{r^2 - 2rxm + x^2m^2 + 4xr_1'm}}{2x}.$$

Since $x \le r/(\sqrt{3}m)$, the inequalities in (9) hold when $\frac{r}{m} \ge \frac{1}{1 - 1/\sqrt{3}}$. □

References

[1] N. Alon and J. Spencer, *The Probabilistic Method*, third edition.

[2] D.W. Cranston, C. Smyth, and D.B. West, Revolutionaries and spies on trees and unicyclic graphs, submitted.

[3] D. Howard and C. Smyth, Revolutionaries and spies, submitted.