Introduction to Mathematical Reason	MATH 300 Test #2	April 10, 2012
Name:____________________	R. Hammack	Score:______

1. (14 points) Suppose A, B and C are sets, and $C \neq \emptyset$. Prove that $A \times C \subseteq B \times C$ if and only if $A \subseteq B$.

Proof. First we will prove that if $A \times C \subseteq B \times C$, then $A \subseteq B$.
We use direct proof. Suppose $A \times C \subseteq B \times C$.

To show $A \subseteq B$, suppose $a \in A$. Since $C \neq \emptyset$, there is an element $c \in C$
Therefore $(a, c) \in A \times C$ by definition of the Cartesian product.
As $A \times C \subseteq B \times C$, it follows that $(a, c) \in B \times C$.
From this, we get $a \in B$ by definition of the Cartesian product.
We've now shown $a \in A$ implies $a \in B$, so $A \subseteq B$.

This completes the proof that if $A \times C \subseteq B \times C$, then $A \subseteq B$.

Conversely, we need to prove that $A \subseteq B$ implies $A \times C \subseteq B \times C$.

We use direct proof. Suppose $A \subseteq C$.

We need to show $A \times C \subseteq B \times C$.
Thus assume $(a, b) \in A \times C$.
Then $a \in A$ and $b \in C$ by definition of the Cartesian product.
Because $A \subseteq B$, it follows from $a \in A$ that also $a \in B$.
Now we have $a \in B$ and $c \in C$, so $(a, c) \in B \times C$.
From this it follows that $A \times C \subseteq B \times C$.

This completes the proof that if $A \subseteq B$, then $A \times C \subseteq B \times C$. ■

2. Suppose A, B, C and D are sets.

(a) (10 points) Prove that $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.

Proof. Suppose $(a, b) \in (A \times B) \cup (C \times D)$.
By definition of union, this means $(a, b) \in (A \times B)$ **or** $(a, b) \in (C \times D)$.
We examine these two cases individually.

Case 1. Suppose $(a, b) \in (A \times B)$. By definition of $\times$, it follows that $a \in A$ and $b \in B$.
From this, it follows from the definition of $\cup$ that $a \in A \cup C$ and $b \in B \cup D$.
Again from the definition of $\times$, we get $(a, b) \in (A \cup C) \times (B \cup D)$.

Case 2. Suppose $(a, b) \in (C \times D)$. By definition of $\times$, it follows that $a \in C$ and $b \in D$.
From this, it follows from the definition of $\cup$ that $a \in A \cup C$ and $b \in B \cup D$.
Again from the definition of $\times$, we get $(a, b) \in (A \cup C) \times (B \cup D)$.

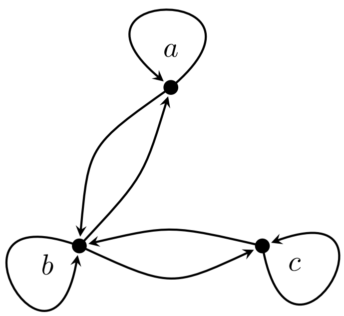
In either case, we obtained $(a, b) \in (A \cup C) \times (B \cup D)$,
so we've proved that $(a, b) \in (A \times B) \cup (C \times D)$ implies $(a, b) \in (A \cup C) \times (B \cup D)$.

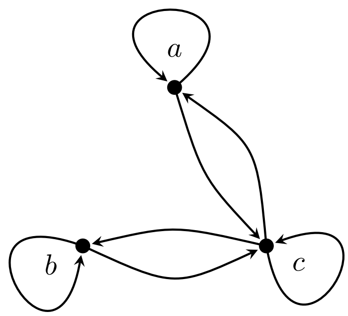
Therefore $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$. ■

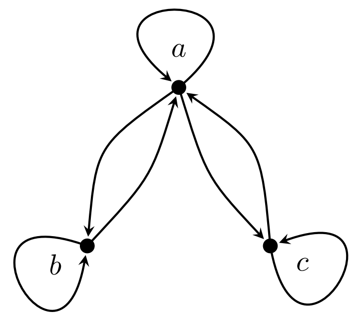
(b) (10 points) Give a counterexample showing $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$ is not always true.

Let $A = \{a\}$, $B = \{b\}$, $C = \{c\}$, and $D = \{d\}$.
Then $(A \times B) \cup (C \times D) = \{(a, b)\} \cup \{(c, d)\} = \{(a, b), (c, d)\}$.
Also, $(A \cup C) \times (B \cup D) = \{a, c\} \times \{b, d\} = \{(a, b), (a, d), (c, b), (c, d)\}$.
Then $(A \times B) \cup (C \times D) \neq (A \cup C) \times (B \cup D)$.

3. (10 points) Draw diagrams for all the different relations on $A = \{a, b, c\}$ that are both reflexive and symmetric, but **not** transitive.







Editorial Comment: In grading the tests, I noticed that there was a tendency to include $R = \{(a, a), (b, b), (c, c)\}$. This relation is actually transitive, for it is impossible for $(xRy) \wedge (yRz) \Longrightarrow xRz$ to ever be false.

4. (14 points) Prove that $3^1 + 3^2 + 3^3 + \cdots + 3^n = \dfrac{3^{n+1} - 3}{2}$ for every $n \in \mathbb{N}$.

Proof (Induction) If $n = 1$, then this statement is simply $3^1 = \frac{3^2-3}{2} = \frac{6}{2} = 3$, which is true.

Now we will show that if the statement is true for some $n = k \geq 1$, then it is true for $n = k + 1$. In other words, we will prove that **if** $3^1 + 3^2 + 3^3 + \cdots + 3^k = \frac{3^{k+1}-3}{2}$, **then** $3^1 + 3^2 + 3^3 + \cdots + 3^k + 3^{k+1} = \frac{3^{(k+1)+1}-3}{2}$. We use direct proof. Assume $3^1 + 3^2 + 3^3 + \cdots + 3^k = \frac{3^{k+1}-3}{2}$. Observe that

$$\begin{aligned}
3^1 + 3^2 + 3^3 + \cdots + 3^k + 3^{k+1} &= (3^1 + 3^2 + 3^3 + \cdots + 3^k) + 3^{k+1} \\
&= \frac{3^{k+1} - 3}{2} + 3^{k+1} \\
&= \frac{3^{k+1} - 3}{2} + \frac{2 \cdot 3^{k+1}}{2} \\
&= \frac{3^{k+1} - 3 + 2 \cdot 3^{k+1}}{2} \\
&= \frac{3 \cdot 3^{k+1} - 3}{2} \\
&= \frac{3^1 \cdot 3^{k+1} - 3}{2} \\
&= \frac{3^{(k+1)+1} - 3}{2}
\end{aligned}$$

We have now established that $3^1 + 3^2 + 3^3 + \cdots + 3^k + 3^{k+1} = \frac{3^{(k+1)+1}-3}{2}$. Thus we have shown that if the statement is true for $n = k$, then it is true for $n = k + 1$. This completes the proof by induction. ■

5. (14 points) Recall that the Fibonacci Sequence is defined as $F_1 = 1$, $F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Use induction to prove that $\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$ for every $n \in \mathbb{N}$.

Proof (Induction)

If $n = 1$, then this statement is simply $\sum_{i=1}^{1} F_i^2 = F_1 F_{1+1}$, that is, $F_1^2 = F_1 F_2$, which is just the (true) statement $1^2 = 1 \cdot 1$.

Now we will show that if the statement is true for some $n = k \geq 1$, then it is true for $n = k+1$. In other words, we will prove that **if** $\sum_{i=1}^{k} F_i^2 = F_k F_{k+1}$, **then** $\sum_{i=1}^{k+1} F_i^2 = F_{k+1} F_{(k+1)+1}$. We use direct proof. Assume $\sum_{i=1}^{k} F_i^2 = F_k F_{k+1}$. Then

$$\begin{aligned}
\sum_{i=1}^{k+1} F_i^2 &= \left(\sum_{i=1}^{k} F_i^2 \right) + F_{k+1}^2 \\
&= F_k F_{k+1} + F_{k+1}^2 \\
&= F_{k+1} \left(F_k + F_{k+1} \right) \\
&= F_{k+1} \left(F_{k+2} \right) \qquad \text{(As } F_k + F_{k+1} = F_{k+2} \text{ by definition of the Fibonacci sequence.)} \\
&= F_{k+1} F_{(k+1)+1}
\end{aligned}$$

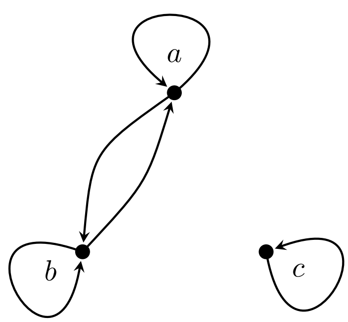
We have now established that $\sum_{i=1}^{k+1} F_i^2 = F_{k+1} F_{(k+1)+1}$. Thus we have shown that if the statement is true for $n = k$, then it is true for $n = k+1$. This completes the proof by induction. ■

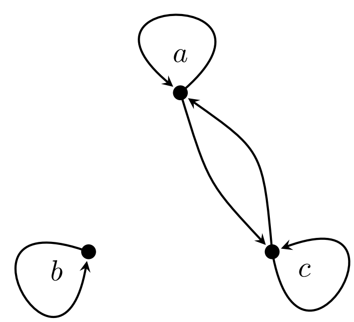
6. (14 points) Prove or disprove:
If A and B are sets, then $\mathscr{P}(A) - \mathscr{P}(B) \subseteq \mathscr{P}(A - B)$.

This is **false.**
Disproof: Here is a counterexample:
Let $A = \{1, 2\}$ and $B = \{1\}$.
Then $\mathscr{P}(A) - \mathscr{P}(B) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\} - \{\emptyset, \{1\}\} = \{\{2\}, \{1, 2\}\}$.
Also $\mathscr{P}(A - B) = \mathscr{P}(\{2\}) = \{\emptyset, \{2\}\}$.
In this example we have $\mathscr{P}(A) - \mathscr{P}(B) \not\subseteq \mathscr{P}(A - B)$.

7. (14 points) Prove or disprove:
Suppose R and S are equivalence relations on a set A. Then $R \cup S$ is also an equivalence relation on A.

This is **false**.
For a counterexample, let R and S be the equivalence relations on $A = \{a, b, c\}$ diagramed below.





$R = \{(a, a), (b, b), (c, c), (a, b), (b, a)\}$ $R = \{(a, a), (b, b), (c, c), (a, c), (c, a)\}$

Both R and S are equivalence relations, because they are reflexive, symmetric and transitive. Their union is the relation $T = R \cup S = \{(a, a), (b, b), (c, c), (a, b), (b, a), (a, c), (c, a)\}$, which is diagramed below. It is not transitive because $(bTa) \wedge (aTc) \Longrightarrow (bTc)$ is false. Therefore the union is not an equivalence relation.