Introduction to Mathematical Reason | MATH 300 Test #2 | April 12, 2011

Name:________________ R. Hammack Score:______

1. (14 points) Prove that $x \in \{12a + 45b : a, b \in \mathbb{Z}\}$ if and only if $3 \mid x$.

   Proof. ($\Longleftarrow$) First we show (with direct proof) that if $x \in \{12a + 45b : a, b \in \mathbb{Z}\}$, then $3 \mid x$.
   Suppose $x \in \{12a + 45b : a, b \in \mathbb{Z}\}$. This means $x = 12a + 45b$ for some integers a and b.
   As $x = 12a + 45b = 3(4a + 15b)$, where $4a + 15b \in \mathbb{Z}$, we see that $3 \mid x$.

   ($\Longrightarrow$) Conversely, we now show (with direct proof) that if $3 \mid x$, then $x \in \{12a + 45b : a, b \in \mathbb{Z}\}$.
   Suppose $3 \mid x$. This means $x = 3k$ for some integer k.
   Observe now that $x = 3k = 48k - 45k = \boxed{12 \cdot (4k) + 45 \cdot (-k)}$.
   Letting a and b be the integers $a = 4k$ and $b = -k$, the above gives $x = 12a + 45b$.
   Therefore $x \in \{12a + 45b : a, b \in \mathbb{Z}\}$. ■

2. (14 points) Suppose A, B, C and D are sets. Prove that $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.

   Proof. Suppose $(x, y) \in (A \times B) \cup (C \times D)$. (We want to show that this implies $(x, y) \in (A \cup C) \times (B \cup D)$.)
   By definition of union, $(x, y) \in (A \times B)$ or $(x, y) \in (C \times D)$.
   We consider these cases separately.

   CASE I: Suppose $(x, y) \in (A \times B)$.
   By definition of the Cartesian product, we have $x \in A$ and $y \in B$.
   By definition of union, it follows that $x \in A \cup C$ and $y \in B \cup D$.
   Again, by definition of the Cartesian product, we get $(x, y) \in (A \cup C) \times (B \cup D)$.

   Thus in this case we have $(x, y) \in (A \times B) \cup (C \times D)$ implies $(x, y) \in (A \cup C) \times (B \cup D)$.

   CASE II: Suppose $(x, y) \in (C \times D)$.
   By definition of the Cartesian product, we have $x \in C$ and $y \in D$.
   By definition of union, it follows that $x \in A \cup C$ and $y \in B \cup D$.
   Again, by definition of the Cartesian product, we get $(x, y) \in (A \cup C) \times (B \cup D)$.

   Thus in this case we have $(x, y) \in (A \times B) \cup (C \times D)$ implies $(x, y) \in (A \cup C) \times (B \cup D)$.

   The above has shown $(x, y) \in (A \times B) \cup (C \times D)$ implies $(x, y) \in (A \cup C) \times (B \cup D)$,
   and therefore $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$. ■

3. (14 points) Prove that $\{3a + 5b : a, b \in \mathbb{Z}\} = \mathbb{Z}$.

Proof. First we will show that $\{3a + 5b : a, b \in \mathbb{Z}\} \subseteq \mathbb{Z}$.
Suppose $x \in \{3a + 5b : a, b \in \mathbb{Z}\}$. This means that $x = 3a + 5b$ for some integers a and b.
Then $x = 3a + 5b$ is an integer, so $x \in \mathbb{Z}$.
This reasoning establishes that $\{3a + 5b : a, b \in \mathbb{Z}\} \subseteq \mathbb{Z}$.

Next we will show that $\mathbb{Z} \subseteq \{3a + 5b : a, b \in \mathbb{Z}\}$.
Suppose $x \in \mathbb{Z}$. Then $x = 3 \cdot (-3x) + 5 \cdot 2x$.
Thus $x = 3a + 5b$, where $a = -3x \in \mathbb{Z}$ and $b = 2x \in \mathbb{Z}$. Consequently $x \in \{3a + 5b : a, b \in \mathbb{Z}\}$.
The above has established that $\mathbb{Z} \subseteq \{3a + 5b : a, b \in \mathbb{Z}\}$.

As $\{3a + 5b : a, b \in \mathbb{Z}\} \subseteq \mathbb{Z}$ and $\mathbb{Z} \subseteq \{3a + 5b : a, b \in \mathbb{Z}\}$, it follows that $\{3a + 5b : a, b \in \mathbb{Z}\} = \mathbb{Z}$ ■

4. (15 points) Recall that Fiboacci Sequence is defined as $F_1 = 1$, $F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$.
Use induction to prove that $F_1^2 + F_2^2 + F_3^2 + F_4^2 + \cdots + F_n^2 = F_n F_{n+1}$.

Proof. (Induction)

First, for the basis step, note that if $n = 1$, then the equation states that $F_1^2 = F_1 F_2$, and this reduces to the (true) statement $1^2 = 1 \cdot 1$. Thus the equation is true when $n = 1$.

(Although it is not necessary, we can also verify the statement for $n = 2$. It is $F_1^2 + F_2^2 = F_2 F_3$, and this reduces to the [true] statement $1^2 + 1^2 = 1 \cdot 2$. Thus the equation is true when $n = 1$.)

Next assume that for some $k \geq 1$ we have $F_1^2 + F_2^2 + F_3^2 + F_4^2 + \cdots + F_n^2 = F_k F_{k+1}$, that is, assume that the given equation is true for $n = k$. In what follows we show that this assumption implies that the equation is true for $n = k + 1$. Observe that

$$\begin{aligned} F_1^2 + F_2^2 + F_3^2 + F_4^2 + \cdots + F_k^2 + F_{k+1}^2 &= \\ (F_1^2 + F_2^2 + F_3^2 + F_4^2 + \cdots + F_k^2) + F_{k+1}^2 &= \\ F_k F_{k+1} + F_{k+1}^2 &= \\ F_{k+1}(F_k + F_{k+1}) &= F_{k+1} F_{k+2}. \end{aligned}$$

(The last step used the Fibonacci property $F_k + F_{k+1} = F_{k+2}$.) The above has established that

$$F_1^2 + F_2^2 + F_3^2 + F_4^2 + \cdots + F_k^2 + F_{k+1}^2 = F_{k+1} F_{k+2},$$

which means that the given equation is true for $n = k + 1$.

This completes the proof by induction. ■

5. (14 points) Use induction to prove that $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots+\frac{n}{(n+1)!} = 1-\frac{1}{(n+1)!}$.

Proof. (Induction)

First, for the basis step, note that if $n=1$, then the equation states that $\frac{1}{2!} = 1-\frac{1}{(1+1)!}$, and this reduces to the (true) statement $\frac{1}{2}=\frac{1}{2}$. Thus the equation is true when $n=1$.

Next assume that for some $k \geq 1$ we have $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots+\frac{k}{(k+1)!} = 1-\frac{1}{(k+1)!}$. (That is, assume the equation is true for $n=k$.) In what follows we show that the equation is true for $n=k+1$. Observe that

$$
\begin{aligned}
\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots+\frac{k}{(k+1)!}+\frac{k+1}{((k+1)+1)!} &= \\
\left(\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots+\frac{k}{(k+1)!}\right)+\frac{k+1}{(k+2)!} &= \\
1-\frac{1}{(k+1)!}+\frac{k+1}{(k+2)!} &= \\
1-\frac{k+2}{k+2}\frac{1}{(k+1)!}+\frac{k+1}{(k+2)!} &= \\
1-\frac{k+2}{(k+2)!}+\frac{k+1}{(k+2)!} &= \\
1+\frac{-(k+2)+(k+1)}{(k+2)!} &= \\
1+\frac{-1}{(k+2)!} &= \\
1-\frac{1}{(k+2)!} &= 1-\frac{1}{((k+1)+1)!}
\end{aligned}
$$

The above has established that

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots+\frac{k}{(k+1)!}+\frac{k+1}{((k+1)+1)!} = 1-\frac{1}{((k+1)+1)!},$$

which means that the given equation is true for $n=k+1$.

This completes the proof by induction. ■

6. (14 points) Prove or disprove:
If A and B are sets, then $\mathscr{P}(A \cup B) = \mathscr{P}(A) \cup \mathscr{P}(B)$.

This is **False**. Here is a counterexample:

Let $A = \{1\}$ and $B = \{2\}$, so that we have

$$\mathscr{P}(A) \cup \mathscr{P}(B) = \{\emptyset, \{1\}\} \cup \{\emptyset, \{2\}\} = \boxed{\{\emptyset, \{1\}, \{2\}\}.} \tag{1}$$

Now observe that

$$\mathscr{P}(A \cup B) = \mathscr{P}(\{1, 2\}) = \boxed{\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}.} \tag{2}$$

Equations (1) and (2) above establish that it is possible that $\mathscr{P}(A \cup B) \neq \mathscr{P}(A) \cup \mathscr{P}(B)$.

7. (15 points) Prove or disprove:
If R and S are two equivalence relations on a set A, then $R \cap S$ is also an equivalence relation on A.

This is **TRUE**. A proof follows.

Proof. Suppose R and S are two equivalence relations on a set A. Then since $R \subseteq A \times A$ and $S \subseteq A \times A$, it follows that $R \cap S \subseteq A \times A$, so $R \cap S$ is a relation on A. We need to check that it is an equivalence relation. For simplicity, set $U = R \cap S$, so we need to check that U is an equivalence relation.

First we show U is reflexive: Suppose $x \in A$. Since R and S are both reflexive relations on A (because they are equivalence relations), it follows that $(x, x) \in R$ and $(x, x) \in S$. Therefore $(x, x) \in R \cap S = U$. This shows $(x, x) \in U$, and hence xUx for every element of A, so U is reflexive.

Next we show U is symmetric: Suppose $x, y \in A$, and xUy. We need to show that this implies yUx. Note that xUy means $(x, y) \in U = R \cap S$. Thus $(x, y) \in R$ and $(x, y) \in S$. Since R and S are both equivalence relations (and hence both symmetric), it follows that $(y, x) \in R$ and $(y, x) \in S$. Therefore $(y, x) \in R \cap S = U$, meaning yUx. We've now proved that xUy implies yUx for any $x, y \in A$, so U is symmetric.

Next we show U is transitive: Suppose $x, y, z \in A$, and xUy and yUz. We need to show that this implies xUz. Observe that xUy and yUz give us

$$(x, y) \in U = R \cap S,$$

$$(y, z) \in U = R \cap S.$$

It follows $(x, y) \in R$ and $(y, z) \in R$, and since R is transitive we also have $(x, z) \in R$.
Also $(x, y) \in S$ and $(y, z) \in S$, and since S is transitive we also have $(x, z) \in S$.
Therefore (x, z) is an element of both R and S, so $(x, z) \in R \cap S = U$. This means xUz. We've now seen that if xUy and yUz, then xUz. Therefore U is transitive.

At this point we've shown that the relation $U = R \cap S$ is reflexive, symmetric and transitive, so it is an equivalence relation. ■