Introduction to Mathematical Reason	Test 1 MATH 300	March 6, 2012

Name: ______________________ R. Hammack Score: ________

Directions: Please answer the questions in the space provided. To get full credit you must show all of your work. Use of calculators and other electronic devices is not allowed on this test.

1. **Short answer.** Write each of the following sets by listing its elements between braces or describing it with a familiar symbol or symbols.

(a) $\bigcap_{n\in\mathbb{N}} \left\{ x \in \mathbb{R} : \frac{-1}{n} \leq x \leq \frac{1}{n} \right\}$ = ... $\boxed{\{0\}}$

(b) $\bigcup_{n\in\mathbb{N}} \left\{ x \in \mathbb{R} : \frac{-1}{n} \leq x \leq \frac{1}{n} \right\}$ = ... $\boxed{[-1,1] = \{x \in \mathbb{R} : -1 \leq x \leq 1\}}$

(c) $\big\{ X \subseteq \{a,b,c,d\} : |\mathscr{P}(X)| = 8 \big\}$ = ... $\boxed{\big\{ \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\} \big\}}$

2. **Short answer.** Write the following sets in set-builder notation.

(a) $\big\{0, 1, 4, 9, 16, 25, 36, \ldots\big\}$ = ... $\boxed{\{n^2 : n \in \mathbb{Z}\}}$

(b) $\left\{ \ldots, \frac{2}{5}, \frac{1}{2}, 0, \frac{1}{4}, \frac{2}{7}, \frac{3}{10}, \frac{4}{13}, \frac{5}{16}, \frac{6}{19}, \ldots \right\}$ = ... $\boxed{\left\{ \frac{n}{1+3n} : n \in \mathbb{Z} \right\}}$

(c) $\big\{ \{2\}, \{2,4\}, \{2,4,6\}, \{2,4,6,8\}, \{2,4,6,8,10\}, \ldots \big\}$ = ... $\boxed{\big\{\{2,4,\ldots,2n\} : n \in \mathbb{N}\big\}}$

3. Are $Q \Rightarrow P$ and $(\sim P) \Rightarrow (Q \wedge \sim Q)$ logically equivalent? Support your answer with a truth table.

P	Q	$Q \Rightarrow P$	$\sim P$	$(Q \wedge \sim Q)$	$(\sim P) \Rightarrow (Q \wedge \sim Q)$
T	T	**T**	F	F	**T**
T	F	**T**	F	F	**T**
F	T	**F**	T	F	**F**
F	F	**T**	T	F	**F**

From the table, we see that the columns for $Q \Rightarrow P$ and $(\sim P) \Rightarrow (Q \wedge \sim Q)$ are not the same. Therefore the two expressions are **not logically equivalent.**

4. This problem concerns the following statement.
P: Given any $x \in \mathbb{R}$, there exists an element $y \in \mathbb{R}$ for which $xy = 1$.

(a) Is the statement P true or false? **Explain**.

It's FALSE, because given $x = 0 \in \mathbb{R}$, there does not exist any real number y for which $xy = 1$.

(b) Form the negation $\sim P$. Write your answer as an English sentence. (The sentence may use mathematical symbols.)

There exists a number a real number x for which $xy \neq 1$ for every real number y.

5. This problem concerns 4-card hands dealt off of a standard 52-card deck. How many 4-card hands are there for which all four cards are of the same suit or all four cards are red?

Let A be the set of all 4-card hands for which all four cards are of the same suit.

Let B be the set of all 4-card hands for which all four cards are red.

The answer we seek is $|A \cup B|$.
Because $|A \cup B| = |A| + |B| - |A \cap B|$, we need to calculate $|A|, |B|$ and $|A \cap B|$.

Given any one suit (say hearts), there are 13 cards of that suit, so the total number of 4-card hands consisting only of hearts is $\binom{13}{4}$. Similarly there are $\binom{13}{4}$ 4-card hands consisting only of diamonds, $\binom{13}{4}$ with only spades, and $\binom{13}{4}$ with only clubs. Thus there are $4\binom{13}{4}$ 4-card hands in which all cards have the same suit. Therefore $\boxed{|A| = 4\binom{13}{4}.}$

As there are 26 red cards, it follows that $\boxed{|B| = \binom{26}{4}.}$

Finally, $A \cap B$ is the set of all 4-card hands in which all four cards are of the same suit **and** they are all red. Reasoning as we did for $|A|$, it follows that $\boxed{|A \cap B| = 2\binom{13}{4}.}$

Therefore our final answer is

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ &= 4\binom{13}{4} + \binom{26}{4} - 2\binom{13}{4} \\ &= 2\binom{13}{4} + \binom{26}{4} \\ &= \boxed{2\tfrac{13!}{4!\,9!} + \tfrac{26!}{4!\,22!}} \end{aligned}$$

6. Suppose $a, b, c \in \mathbb{Z}$. Prove that if $a \mid b$ and $a \mid (b^2 + c)$, then $a \mid c$. (Hint: Try direct.)

Proof (Direct) Suppose that $a \mid b$ and $a \mid (b^2 + c)$
As $a \mid b$, the definition of divides guarantees that $\boxed{b = ax}$ for some integer x.
Likewise, as $a \mid (b^2 + c)$, the definition of divides guarantees that $\boxed{b^2 + c = ay}$ for some integer y.
Substituting the first boxed equation into the second, we get $(ax)^2 + c = ay$, which is $a^2x^2 + c = ay$.
Transposing this yields $c = ay - a^2x^2 = a(y - ax^2)$.
Therefore $c = a(y - ax^2)$, where $y - ax^2 \in \mathbb{Z}$.
The definition of divides now implies $a \mid c$. ■

7. Suppose $a, b, c \in \mathbb{Z}$. Prove that if $a \nmid bc$, then $a \nmid b$ and $a \nmid c$. (Hint: Try contrapositive.)

Proof (Contrapositive)
Suppose it is not the case that $a \nmid b$ and $a \nmid c$.
Then (Using DeMorgan's Law) $a \mid b$ or $a \mid c$.
We now break into cases according to whether $a \mid b$ or $a \mid c$.

CASE 1 Suppose $a \mid b$
Then $b = ax$ for some integer x, by definition of divides.
From this, $bc = a(cx)$, where $cx \in \mathbb{Z}$, which means $a \mid bc$, by definition of divides.

CASE 2 Suppose $a \mid c$
Then $c = ax$ for some integer x, by definition of divides.
From this $bc = bax = a(bx)$, where $bx \in \mathbb{Z}$, which means $a \mid bc$, by definition of divides.

The above cases show that whether $a \mid b$ or $a \mid c$, it is true that $a \mid bc$.
Therefore it is not the case that $a \nmid bc$. ■

Editorial Comment: A third case under which $a \mid b$ and $a \mid c$ is not necessary, because in this event either Case 1 or 2 above applies.

8. Prove that $\sqrt{6}$ is irrational. (Hint: Try contradiction.)

Proof Suppose for the sake of contradiction that $\sqrt{6}$ is not irrational, that is, that it is rational.
Then $\sqrt{6} = \frac{a}{b}$ for some integers a and b.
We may assume that this fraction is fully reduced; In particular $\boxed{a \text{ and } b \text{ are not both even.}}$
From the above equation we get $b\sqrt{6} = a$. Squaring both sides, $6b^2 = a^2$.
Therefore $a^2 = 2(3b^2)$; it follows that a^2 is even, so $\boxed{a \text{ is even.}}$
Thus $a = 2m$ for some integer m.
Substituting this into $6b^2 = a^2$ gives $6b^2 = 4m^2$, or $3b^2 = 2m^2$
From $3b^2 = 2m^2$ it follows that $3b^2$ is even.
Then $\boxed{b \text{ is even too,}}$ for otherwise b^2 (hence also $3b^2$) would be odd.
The boxed terms above imply that a and b are not both even **and** a and b are both even.
This contradiction proves the theorem. ■

9. Suppose $x, y \in \mathbb{R}$. Prove that if $xy - x^2 + x^3 \geq x^2y^3 + 4$, then $x \geq 0$ or $y \leq 0$.

Proof (Contrapositive) Suppose it is not the case that $x \geq 0$ or $y \leq 0$.
Then DeMorgan's Laws give $x < 0$ **and** $y > 0$.
Therefore x is negative and y is positive.
Thus $xy - x^2 + x^3$ is negative (all its terms are negative) and $x^2y^3 + 4$ is positive.
From this, $xy - x^2 + x^3 < x^2y^3 + 4$.
Therefore it is not the case that $xy - x^2 + x^3 \geq x^2y^3 + 4$. ■

10. Suppose $a, b, c \in \mathbb{Z}$, and $n \in \mathbb{N}$. Prove the following:
If $a \equiv b \pmod{n}$ and $a \equiv c \pmod{n}$, then $2a \equiv b + c \pmod{n}$.

Proof (Direct) Suppose $a \equiv b \pmod{n}$ and $a \equiv c \pmod{n}$.
By definition of congruence modulo n, we have $n \mid (a - b)$ and $n \mid (a - c)$.
By definition of divides, this gives $a - b = nk$ and $a - c = n\ell$ for some integers k and ℓ.
Adding $a - b = nk$ to $a - c = n\ell$ gives $a - b + a - c = nk + n\ell$.
Simplifying this produces $2a - (b + c) = n(k + \ell)$.
Since $k + \ell \in \mathbb{Z}$, it follows from the definition of divides that $n \mid (2a - (b + c))$.
Finally, from this the definition of congruence modulo n implies $2a \equiv b + c \pmod{n}$. ■