Introduction to Mathematical Reason | Test 1 MATH 300 | March 8, 2011

Name: ______________________ R. Hammack Score: ________

Directions: Please answer the questions in the space provided. To get full credit you must show all of your work. Use of calculators and other electronic devices is not allowed on this test.

1. **Short answer.** Write each of the following sets by listing its elements between braces or describing it with a familiar symbol or symbols.

   (a) $\{n \in \mathbb{Z} : n^2 \leq 4\}$ = ... $\boxed{\{-2,-1,0,1,2\}}$

   (b) $\{x : (x,x) \in \mathbb{Z} \times \mathbb{N}\}$ = ... $\boxed{\mathbb{N}}$

   (c) $\displaystyle\bigcap_{X \in \mathscr{P}(\mathbb{N})} \big(X \cup \{0\}\big)$ = ... $\boxed{\{0\}}$

   (d) $\mathscr{P}(\{1,2\}) - \mathscr{P}(\{2\})$ = ... $\boxed{\{\{1\},\{1,2\}\}}$

   (e) $\{X \subseteq \mathbb{R} : |X \cup \{0,1\}| = 2\}$ = ... $\boxed{\{\emptyset,\{0\},\{1\},\{0,1\}\}}$

2. **Short answer.** Write the following sets in set-builder notation.

   (a) $\{5,\ 8,\ 11,\ 14,\ 17,\ 20,\ldots\}$ = ... $\boxed{\{3n+2 : n \in \mathbb{N}\}}$

   (b) $\left\{\ldots,\frac{1}{16},\frac{1}{8},\frac{1}{4},\frac{1}{2},1,2,4,8,16,\ldots\right\}$ = ... $\boxed{\{2^n : n \in \mathbb{Z}\}}$

   (c) $\{\ldots,\ (-3,-4),\ (-2,-2),\ (-1,0),\ (0,2),\ (1,4),\ (2,6),\ (3,8),\ \ldots\}$ = .. $\boxed{\{(n,2n+2) : n \in \mathbb{Z}\}}$

3. Are $P \Rightarrow Q$ and $\sim (P \vee \sim Q)$ logically equivalent? Support your answer with a truth table.

P	Q	$P \Rightarrow Q$	$P \wedge \sim Q$	$\sim (P \wedge \sim Q)$
T	T	**T**	F	**T**
T	F	**F**	T	**F**
F	T	**T**	F	**T**
F	F	**T**	F	**T**

Since the columns for $P \Rightarrow Q$ and $\sim (P \wedge \sim Q)$ agree, we conclude that they are logically equivalent

4. This problem concerns the following statement.
$P:$ Given any subset X of $\mathbb{R}$, there exists a subset Y of $\mathbb{R}$ for which $X \cap Y = \emptyset$ and $X \cup Y = \mathbb{R}$.

   (a) Is the statement P true or false? **Explain**.

   It is TRUE. Given any subset X of $\mathbb{R}$, let $Y = \overline{X}$, which is another subset of $\mathbb{R}$. Notice that we do indeed have $X \cap Y = \emptyset$ and $X \cup Y = \mathbb{R}$.

   (b) Form the negation $\sim P$. Write your answer as an English sentence (which may use mathematical symbols).

   Symbolically, the sentence is:
   $\forall X \subseteq \mathbb{R}, \exists Y \subset \mathbb{R}, (X \cap Y = \emptyset) \wedge (X \cup Y = \mathbb{R})$
   Forming the negation, we get

$$
\begin{aligned}
\sim \big(\forall X \subseteq \mathbb{R}, \exists Y \subset \mathbb{R}, (X \cap Y = \emptyset) \wedge (X \cup Y = \mathbb{R})\big) &= \\
\exists X \subseteq \mathbb{R}, \sim \big(\exists Y \subset \mathbb{R}, (X \cap Y = \emptyset) \wedge (X \cup Y = \mathbb{R})\big) &= \\
\exists X \subseteq \mathbb{R}, \forall Y \subset \mathbb{R}, \sim \big((X \cap Y = \emptyset) \wedge (X \cup Y = \mathbb{R})\big) &= \\
\exists X \subseteq \mathbb{R}, \forall Y \subset \mathbb{R}, \sim (X \cap Y = \emptyset) \vee \sim (X \cup Y = \mathbb{R}) &= \\
\exists X \subseteq \mathbb{R}, \forall Y \subset \mathbb{R}, (X \cap Y \neq \emptyset) \vee (X \cup Y \neq \mathbb{R})
\end{aligned}
$$

Thus the negation is as follows.

There exists a subset $X \subseteq \mathbb{R}$ with the property that $X \cap Y \neq \emptyset$ or $X \cup Y \neq \mathbb{R}$ for each $Y \subset \mathbb{R}$.

5. This question concerns lists **<u>of length 5</u>** made from the symbols A, B, C, D, E, F, G, H, **<u>without repetition</u>**.

(a) How many such lists are possible?

Answer: $\boxed{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}$.

(b) How many such lists are there that are **not** in alphabetical order?

First, think about how many such lists are in alphabetical order. To make any such list, select any five of the eight letters, and arrange them in order. There are $\binom{8}{5}$ ways to do this.

Now, the number of lists that **are not** in alphabetical order equals the total number of all lists (i.e., the answer to part (a)) minus the number of those that are not in alphabetical order.

Answer: $\boxed{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 - \binom{8}{5}.}$

6. Prove: Suppose $a, b, c \in \mathbb{Z}$. If $a^2 \mid b$ and $b^3 \mid c$, then $a^6 \mid c$. (Hint: Try direct.)

Proof (Direct) Suppose $a^2 \mid b$ and $b^3 \mid c$.
By definition of divides, this means that $b = a^2 c$ and $c = b^3 d$ for some integers c and d.
Plugging the value $b = a^2 c$ into $c = b^3 d$, we get $c = (a^2 c)^3 d = a^6 (c^3 d)$.
Now we have obtained $c = a^6 (c^3 d)$, or rather $c = a^6 k$, where $k = c^3 d \in \mathbb{Z}$.
Again, by the definition of divides, this yields $a^6 \mid c$ ■

7. Prove: Suppose $a \in \mathbb{Z}$. If a^3 is not divisible by 8, then a is odd. (Hint: Try contrapositive.)

   Proof (Contrapositive) Suppose that a is not odd, that is, that a is even.
   Then $a = 2c$, for some intger c, by definition of an even integer.
   Consequently, $a^3 = (2c)^3 = 8c^3$.
   Therefore $a^3 = 8d$ for the integer $d = c^3$.
   By definition of divides, this implies that $8|a^3$, so it is not true that a^3 is not divisible by 8. ■

8. Prove: If $a \in \mathbb{Z}$, then $4 \nmid (a^2 - 3)$. (Hint: Try contradiction.)

   Proof Suppose for the sake of contradiction that a is an integer and $4 \mid (a^2 - 3)$.
   By definition of divides, this means there is an integer c for which

$$a^2 - 3 = 4c \tag{1}$$

   Then $a^2 = 4c - 3 = 4c - 4 + 1 = 2(2c - 2) + 1$, where $2c - 2$ is an integer; this means a^2 is odd.
   We have previously shown that if a^2 is odd, then a is odd, so we now know that a is odd.
   Thus $a = 2d + 1$ for some integer d. Now we will plug $a = 2d + 1$ into Equation 1, to get

$$\begin{aligned}
a^2 - 3 &= 4c \\
(2d+1)^2 - 3 &= 4c \\
4d^2 + 4d + 1 - 3 &= 4c \\
4d^2 + 4d - 2 &= 4c \\
\frac{1}{2}(4d^2 + 4d - 2) &= \frac{1}{2}4c \\
2d^2 + 2d - 1 &= 2c \\
2d^2 + 2d - 2c &= 1 \\
2(d^2 + d - c) &= 1
\end{aligned}$$

   As $e = d^2 + d - c$ is an integer, we infer from the above that $1 = 2e$ is even. This is a contradiction.
   ■

9. Prove: If $a, b \in \mathbb{Z}$ and $a + b$ is even, then $a^2 + b^2$ is even. (Hint: Try direct.)

Proof (Direct) Suppose that $a + b$ is even.
This means $a + b = 2k$, for some integer k.
Then $(a + b)^2 = (2k)^2$, and this becomes $a^2 + 2ab + b^2 = 4k^2$.
Transposing the above equation, we obtain $a^2 + b^2 = 4k^2 - 2ab = 2(2k^2 - ab)$.
Therefore, we have $a^2 + b^2 = 2(c)$, where $c = 2k^2 - ab \in \mathbb{Z}$.
Thus, by the definition of an even integer, $a^2 + b^2$ is even. ■

10. Suppose $a, b, c \in \mathbb{Z}$, and $n \in \mathbb{N}$. Prove the following:
If $a \equiv b \pmod{n}$ and $b^2 \equiv c^2 \pmod{n}$, then $a^2 \equiv c^2 \pmod{n}$.

Proof (Direct) Suppose $a \equiv b \pmod{n}$ and $b^2 \equiv c^2 \pmod{n}$.
By definition of congruence modulo n, this means that $n \mid (a - b)$ and $n \mid (b^2 - c^2)$.
By definition of divides, we have $a - b = nk$ and $b^2 - c^2 = n\ell$ for some integers k and ℓ.
Multiplying both sides of the first by $(a + b)$, we have $(a - b)(a + b) = nk(a + b)$ and $b^2 - c^2 = n\ell$.
Thus we have $a^2 - b^2 = nk(a + b)$ and $b^2 - c^2 = n\ell$.
Adding these two equations results in $(a^2 - b^2) + (b^2 - c^2) = nk(a + b) + n\ell$.
Simplifying, we get $a^2 - c^2 = n(ka + kb + \ell)$, where $ka + kb + \ell \in \mathbb{Z}$.
By definition of divides, this implies $n \mid (a^2 - c^2)$.
By definition of congruence modulo n, we have $a^2 \equiv c^2 \pmod{n}$. ■