Introduction to Mathematical Reason	Test 1 (Sample) MATH 300	March 4, 2011

Name: ____________________ R. Hammack Score: ________

Directions: Please answer the questions in the space provided. To get full credit you must show all of your work. Use of calculators and other computing or communication devices is not allowed on this test.

1. **Short answer.** Write each of the following sets by listing its elements or describing it with a familiar symbol or symbols.

   (a) $\{n \in \mathbb{Z} : |n| \le 2\} \;=\; \boxed{\{-2,-1,0,1,2\}}$

   (b) $\{x \in \mathbb{R} \,:\, \cos(\pi x) = -1\} \;=\; \boxed{\{\ldots,-5,-3,-1,1,3,5,\ldots\}} = \boxed{\{2n+1 : n \in \mathbb{Z}\}}$

   (c) $\{X \in \mathscr{P}(\mathbb{N}) \,:\, X \cap \{1,2\} = X\} \;=\; \boxed{\{\{\},\{1\},\{2\},\{1,2\}\}}$

   (d) $\displaystyle\bigcap_{n\in\mathbb{N}} [1, 2+\tfrac{1}{n}] \;=\; \boxed{[1,2]} = \boxed{\{x \in \mathbb{R} : 1 \le x \le 2\}}$

   (e) $\mathscr{P}(\{1\}) \times \mathscr{P}(\{2\}) \;=\; \boxed{\{\, (\emptyset,\emptyset),\ (\emptyset,\{2\}),\ (\{1\},\emptyset),\ (\{1\},\{2\})\,\}}$

2. **Short answer.** Write the following sets in set-builder notation.

   (a) $\{2,-7,12,-17,22,-27,\ldots\} = \boxed{\big\{(-1)^{n+1}(5n-3) : n \in \mathbb{N}\big\}}$

   (b) $\left\{\frac{1}{1},\frac{2}{3},\frac{3}{9},\frac{4}{27},\frac{5}{81},\ldots\right\} = \boxed{\left\{\frac{n}{3^{n-1}} : n \in \mathbb{N}\right\}}$

3. Write a truth table to decide if $(\sim P) \Rightarrow Q$ and $(P \wedge Q) \Rightarrow P$ are logically equivalent.

P	Q	$\sim P$	$P \wedge Q$	$(\sim P) \Rightarrow Q$	$(P \wedge Q) \Rightarrow P$
T	T	F	T	**T**	**T**
T	F	F	F	**T**	**T**
F	T	T	F	**T**	**T**
F	F	T	F	**F**	**T**

Statements $(\sim P) \Rightarrow Q$ and $(P \wedge Q) \Rightarrow P$ are **NOT** logically equivalent because their truth tables do not match up.

4. This problem concerns the following statement.
P : For every subset X of $\mathbb{N}$, there is an integer m for which $|X| = m$.

(a) Is the statement P true or false? Explain.

P says that no matter which set $X \subseteq \mathbb{N}$ you might pick,
there will always be an integer m (depending on X) for which $|X| = m$.

The statement is **FALSE.**
Notice that the set $X = \{2, 4, 6, 8, \ldots\}$ of even numbers is a subset of $\mathbb{N}$,
but there is no integer m for which $|X| = m$, because X is infinite.

Therefore it is untrue that for every subset X of $\mathbb{N}$, there is an integer m for which $|X| = m$.

(b) Form the negation $\sim P$. Write your answer as an English sentence.

Symbolically, the statement P is $\forall X \subseteq \mathbb{N}, \exists m \in \mathbb{Z}, |X| = m$
Its negation is

$$\begin{aligned} \sim P &= \sim (\forall X \subseteq \mathbb{N}, \exists m \in \mathbb{Z}, |X| = m) \\ &= \exists X \subseteq \mathbb{N}, \sim (\exists m \in \mathbb{Z}, |X| = m) \\ &= \exists X \subseteq \mathbb{N}, \forall m \in \mathbb{Z}, \sim (|X| = m) \\ &= \exists X \subseteq \mathbb{N}, \forall m \in \mathbb{Z}, |X| \neq m \end{aligned}$$

Translating back into words, the negation is:
There is a subset X of $\mathbb{N}$ for which for every integer m we have $|X| \neq m$.

Putting this into a more natural form, the translation is
$\boxed{\text{There is a subset } X \text{ of } \mathbb{N} \text{ for which } |X| \neq m \text{ for every integer } m.}$

5. Suppose that $(R \Rightarrow S) \vee \sim (P \wedge Q)$ is **false**.
Is there enough information to determine the truth values of P, Q, R and S? If so, what are they?
(This is most easily done without a truth table.)

For this to be false, both $R \Rightarrow S$ and $\sim (P \wedge Q)$ must be false.
The only way that $R \Rightarrow S$ can be false is if $\boxed{R \text{ is true and } S \text{ is false.}}$
The only way that $\sim (P \wedge Q)$ can be false is if $P \wedge Q$ is true.
The only way that $P \wedge Q$ can be true is if $\boxed{P \text{ and } Q \text{ are both true}}$.

Therefore $\boxed{R, P \text{ and } Q \text{ are all true, and } S \text{ is false.}}$

6. How many 10-digit integers contain no 0's and exactly three 6's?

One solution is as follows. You have a list of 10 blank slots to fill. First select three of the slots for the 6's. There are $\binom{10}{3}$ ways to do this. Once these three slots have been filled with 6's, we need to fill in the other seven slots. Each one can be filled with any digit other than 0 or 6. As there are eight such digits, there are 8^7 ways that the remaining slots can be filled.

This gives a total of $\binom{10}{3}8^7 = \frac{10!}{3!(10-3)!}8^7 = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2}8^7 = (5 \cdot 3 \cdot 8)8^7 =$
$\boxed{15 \cdot 8^8 = 251658240 \text{ possible 10-digit integers with no 0's and exactly three 6's.}}$

Note: Given that you are not using a calculator, it's OK to leave it as $15 \cdot 8^8$, or something of that form.

7. Suppose $a, b, c, d \in \mathbb{Z}$ and $n \in \mathbb{N}$. Prove that if $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $ac \equiv bd \pmod{n}$. (Suggestion: Try direct proof.)

   Proof. (Direct) Suppose $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$.
   By definition of congruence modulo n, this means $n|(a-b)$ and $n|(c-d)$.
   By definition of divisibility, $a - b = nk$ and $c - d = n\ell$ for some $k, \ell \in \mathbb{Z}$.
   Therefore we have $a = b + nk$ and $c = d + n\ell$. Consequently,

$$\begin{aligned} ac &= (b+nk)(d+n\ell) \\ ac &= bd + bn\ell + nkd + n^2k\ell \\ ac - bd &= bn\ell + nkd + n^2k\ell \\ ac - bd &= n(b\ell + kd + nk\ell). \end{aligned}$$

   Since $b\ell + kd + nk\ell \in \mathbb{Z}$, it follows from the above equation that $n|(ac - bd)$.
   This means that $ac \equiv bd \pmod{n}$. ■

8. Suppose $a, b \in \mathbb{Z}$. If $a^2(b^2 - 2b)$ is odd, then both a and b are odd. (Suggestion: Try contrapositive proof.)

   Proof. (Contrapositive) Suppose it is not the case that a and b are odd.
   Then, by DeMorgan's Law, a is even or b is even. Let us look at these cases separately.

   Case 1. Suppose a is even. Then $a = 2c$ for some integer c.
   Thus $a^2(b^2 - 2b) = (2c)^2(b^2 - 2b) = 2(2c^2(b^2 - 2b))$, which is even.

   Case 2. Suppose b is even. Then $b = 2c$ for some integer c.
   Thus $a^2(b^2 - 2b) = a^2((2c)^2 - 2(2c)) = 2(a^2(2c^2 - 2c))$, which is even.

   Thus in either case $a^2(b^2 - 2b)$ is even, so it is not odd. ■

   (NOTE: A third case where both a and b are even is not necessary.
   In that case a is even, a scenario addressed in Case 1.)

9. Prove: For all integers a and b, $a^2 - 4b - 2 \neq 0$.
(Suggestion: Contradiction may be easiest.)

Proof. Suppose for the sake of contradiction that there exist $a, b \in \mathbb{Z}$ for which $a^2 - 4b - 2 = 0$.
Then we have $a^2 = 4b + 2 = 2(2b+1)$, which means a^2 is even.
Therefore a is even also, so $a = 2c$ for some integer c. Plugging this back into $a^2 - 4b - 3 = 0$ gives us

$$\begin{aligned} (2c)^2 - 4b - 2 &= 0 \\ 4c^2 - 4b - 2 &= 0 \\ 4c^2 - 4b &= 2 \\ 2c^2 - 2b &= 1 \\ 2(c^2 - b) &= 1 \end{aligned}$$

From this last equation, we conclude that 1 is an even number, a contradiction. ■

10. Suppose $a, b \in \mathbb{Z}$. If $25 \nmid ab$, then $5 \nmid a$ or $5 \nmid b$.

Proof. (Contrapositive) Suppose it is not the case that $5 \nmid a$ or $5 \nmid b$.
Then (using DeMorgan's Law), we have $5 \mid a$ **and** $5 \mid b$.
By definition of divisibility, this gives $a = 5k$ and $b = 5\ell$ for some $k, \ell \in \mathbb{Z}$.
Then $ab = (5k)(5\ell) = 25k\ell$, that is, $ab = 25c$ for $c = k\ell \in \mathbb{Z}$.
Therefore $25 \mid ab$, by definition of divisibility.
Thus it is not the case that $25 \nmid ab$. ■