# MATH 300 Final Exam Study Guide

- The final exam is scheduled for 1:00–3:50 PM on Thursday, May 3 in our usual classroom.
- The exam will have 20 to 25 questions. Most will be proofs.
- Questions will be roughly evenly distributed from Chapters 1–13.
- There is a sample final exam (blank and solved) on my web page.
- In writing the exam, I assume that on average you have invested about six hours per week (outside of class) on this material, for the entire semester.
- You will be expected to prove things straight from the definitions. (So you will need to know all the definitions listed below.) Many of the questions will be taken from the exercises (both even and odd) or examples in the text. You should know the following material.

1. **Background material.** You should be quite familiar with the material in Chapters 1, 2 and 3. Though there will be only a few exam questions directly related to this material, understanding it is a prerequisite for doing much of the rest of the exam.

2. **Important sets.** You'll need to understand what the following sets are.
   (a) $\emptyset = \{\}$
   (b) $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ (natural numbers)
   (c) $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \ldots\}$ (integers)
   (d) $\mathbb{Q} = \{\frac{a}{b} \ : \ a, b \in \mathbb{Z}, b \neq 0\}$ (rational numbers)
   (e) $\mathbb{R}$ (real numbers)
   (f) Intervals on $\mathbb{R}$: $(a, b) = \{x \in \mathbb{R} \ : \ a < x < b\}$ and $[a, b] = \{x \in \mathbb{R} \ : \ a \leq x \leq b\}$, etc.
   (g) $\mathbb{Z}_n = \{[0], [1], [2], \ldots, [n-1]\}$ (integers modulo $n$)

3. **Definitions** (Not only should you know all these definitions, you should know how to *use* them.)
   (a) An integer $n$ is *even* if $n = 2k$ for some $k \in \mathbb{Z}$.
   (b) An integer $n$ is *odd* if $n = 2k + 1$ for some $k \in \mathbb{Z}$.
   (c) A real number $r$ is *rational* if $r = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$;
   it is *irrational* if there are no integers $a, b \in \mathbb{Z}$ with $r = \frac{a}{b}$.
   (d) If $a, b \in \mathbb{Z}$, then $a | b$ if there is an integer $k \in \mathbb{Z}$ with $b = ka$.
   (e) If $a, b, n \in \mathbb{Z}$ and $n \geq 2$, then $a \equiv b \pmod{n}$ if $n | (a - b)$.
   (f) Cartesian product of sets: $A \times B = \{(x, y) \ : \ x \in A, y \in B\}$.
   (g) Power set: $\mathscr{P}(A) = \{X \ : \ X \subseteq A\}$.
   (h) A *relation on a set* $A$ is a subset $R \subseteq A \times A$; The statement $(a, b) \in R$ is expressed as $aRb$.
      i. $R$ is *reflexive* if $xRx$ for every $x \in A$.
      ii. $R$ is *symmetric* if $xRy \Rightarrow yRx$ for all $x, y \in A$.
      iii. $R$ is *transitive* if $(xRy \wedge yRz) \Rightarrow xRz$ for every $x, y, z \in A$.
      iv. $R$ is an *equivalence relation* if it is reflexive, symmetric and transitive.
      v. If $R$ is an equivalence relation and $a \in A$, the *equivalence class containing* $a$ is $[a] = \{x \in A : aRx\}$.

   (i) A *relation from a set* $A$ *to a set* $B$ is a subset $R \subseteq A \times B$; The statement $(a, b) \in R$ is expressed $aRb$.
   (j) A function $f : A \to B$ is a relation $f \subseteq A \times B$ where for each $a \in A$ there is exactly one $(a, x) \in f$;
   $(a, b) \in f$ is expressed $f(a) = b$; The domain of $f$ is $A$; the codomain is $B$; the *range* is $\{f(a) : a \in A\}$
      i. $f : A \to B$ is *one-to-one* or *injective* if $f(x) = f(y) \Rightarrow x = y$ (or if $x \neq y \Rightarrow f(x) \neq f(y)$).
      ii. $f : A \to B$ is *onto* or *surjective* if for any $b \in B$ there is an $a \in A$ with $f(a) = b$.
      iii. $f : A \to B$ is *bijective* if is both injective and surjective.
   (k) Any bijective function $f : A \to B$ has an *inverse* $f^{-1} : B \to A$, which is a function satisfying $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$ for all $x \in A$.
   (l) Given any function $f : A \to B$,
      i. if $X \subseteq A$, then the *image* of $X$ is $f(X) = \{f(x) : x \in A\}$;
      ii. if $Y \subseteq B$, then the *preimage* of $Y$ is $f^{-1}(Y) = \{x \in A : f(x) \in Y\}$.

   (m) Sets $A$ and $B$ have the *same cardinality* (written $|A| = |B|$) if there is a bijective function $f : A \to B$.
   (n) A set $A$ is *countably infinite* if $|A| = |\mathbb{N}|$.
   (o) A set $A$ is *uncountable* if it is not countably infinite.
   (p) $|A| < |B|$ means there exists an injective function $f : A \to B$ but no bijective function $f : A \to B$.

4. **Important Concepts**
   (a) Multiplication principle
   (b) Direct proof
   (c) Contrapositive proof
   (d) Proof by contradiction
   (e) If-and-only-if proof
   (f) Effective use of cases
   (g) Proof of existence statements
   (h) Mathematical Induction (Smallest counterexample are optional)
   (i) Disproof and counterexamples
5. **Important Techniques**
   (a) Using the multiplication principle
   (b) Prove that $A \subseteq B$ (where $A$ and $B$ are sets)
   (c) Prove that $A = B$ (where $A$ and $B$ are sets)
   (d) Show that a relation on a set is an equivalence relation, and find the equivalence classes.
   (e) Prove that a function is injective, surjective and/or bijective
   (f) Find the inverse of a bijective function
   (g) Compose functions (i.e. work out the composition of two or more functions)
   (h) Prove that two sets have the same (or different) cardinality
   (i) Be able to show a set is countably infinite or uncountable
6. **Important Facts**
   (a) The equivalence classes of an equivalence relation on a set $A$ form a partition of $A$
   (b) The sets $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are countably infinite. The real numbers $\mathbb{R}$ are uncountable.