Additional Induction Problems

1. If n is a natural number, then $0^2 + 1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

   Proof. (By induction)

   Basis Step. If $n = 0$, then the statement is $0^2 = \frac{0(0+1)(2\cdot 0+1)}{6}$, or just $0 = 0$.

   Inductive Step. Assume that for $k \geq 0$, the following is a true statement:
   $0^2 + 1^2 + 2^2 + 3^2 + 4^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$.
   Add $(k+1)^2$ to both sides to get
   $0^2 + 1^2 + 2^2 + 3^2 + 4^2 + \cdots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$
   $0^2 + 1^2 + 2^2 + 3^2 + 4^2 + \cdots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6}$
   $0^2 + 1^2 + 2^2 + 3^2 + 4^2 + \cdots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)+6(k+1)^2}{6}$
   $0^2 + 1^2 + 2^2 + 3^2 + 4^2 + \cdots + k^2 + (k+1)^2 = \frac{(k+1)[k(2k+1)+6(k+1)]}{6}$
   $0^2 + 1^2 + 2^2 + 3^2 + 4^2 + \cdots + k^2 + (k+1)^2 = \frac{(k+1)[2k^2+k+6k+6]}{6}$
   $0^2 + 1^2 + 2^2 + 3^2 + 4^2 + \cdots + k^2 + (k+1)^2 = \frac{(k+1)[2k^2+7k+6]}{6}$
   $0^2 + 1^2 + 2^2 + 3^2 + 4^2 + \cdots + k^2 + (k+1)^2 = \frac{(k+1)[(k+2)(2k+3)]}{6}$
   $0^2 + 1^2 + 2^2 + 3^2 + 4^2 + \cdots + k^2 + (k+1)^2 = \frac{(k+1)[(k+1+1)(2(k+1)+1)]}{6}$
   Thus the statement is true for $k+1$, and this proves the theorem.

2. If n is a natural number, then $9 \mid \left(n^3 + (n+1)^3 + (n+2)^3\right)$.

   Proof. (By induction.)

   Basis Step. If $n = 0$, then the statement is $9 \mid \left(0^3 + (0+1)^3 + (0+2)^3\right)$, which is the true statement $9|9$.

   Inductive Step. Assume that for $k \geq 0$, the statement $9 \mid \left(k^3 + (k+1)^3 + (k+2)^3\right)$ is true. We want to show that $9 \mid \left((k+1)^3 + (k+1+1)^3 + (k+1+2)^3\right)$.
   Now, $(k+1)^3 + (k+1+1)^3 + (k+1+2)^3$
   $= (k+1)^3 + (k+2)^3 + (k+3)^3$
   $=(k+1)^3 + (k+2)^3 + (k^3 + 3k^2 3 + 3k3^2 + 3^3)$
   $= (k+1)^3 + (k+2)^3 + (k^3 + 9k^2 + 27k + 27)$
   $= (k^3 + (k+1)^3 + (k+2)^3) + (9k^2 + 27k + 27)$
   $= (k^3 + (k+1)^3 + (k+2)^3) + 9(k^2 + 3k + 3)$.

   Look at the above expression. It is the sum of two terms, $(k^3 + (k+1)^3 + (k+2)^3)$ and $9(k^2 + 3k + 3)$. By the inductive hypothesis, 9 divides the first term, and clearly 9 divides the second, by definition of divides. Thus it follows that 9 must divide the sum. This means $9 \mid \left((k+1)^3 + (k+1+1)^3 + (k+1+2)^3\right)$.

3. If n is a natural number, then $\Sigma_{k=0}^{n} F_k = F_{n+2} - 1$

   Proof (by induction).

   Basis Step. If $n = 0$, then the statement is $\Sigma_{k=0}^{0} F_k = F_{0+2} - 1$, which is the statement $F_0 = F_{0+2} - 1$, or $F_0 = F_2 - 1$. Since $F_0 = 1$ and $F_2 = 2$, you can see that this is a true statement.

   Inductive Step. Suppose that $\Sigma_{k=0}^{k} F_k = F_{k+2} - 1$ for some natural number k. We want to show that $\Sigma_{k=0}^{k+1} F_k = F_{k+1+2} - 1$. Begin with the equation $\Sigma_{k=0}^{k} F_k = F_{k+2} - 1$, and add an F_{k+1} to both sides. Then you get the equation $\Sigma_{k=0}^{k+1} F_k = F_{k+2} + F_{k+1} - 1$, Now use the fact that $F_{k+2} + F_{k+1} = F_{k+3}$, and we get $\Sigma_{k=0}^{k+1} F_k = F_{k+3} - 1$. This proves the Theorem.

4. If n is a natural number, then $\Sigma_{k=0}^{n} {F_k}^2 = F_n F_{n+1}$.

   Proof (by induction).
   Basis Step. If $n = 0$, then the statement is $\Sigma_{k=0}^{0} {F_k}^2 = F_0 F_{0+1}$, which is the statement ${F_0}^2 = F_0 F_1$. Since $F_0 = 1 = F_1$, this is just the true statement $1^2 = (1)(1)$.

   Inductive Step. Suppose that $\Sigma_{k=0}^{k} {F_k}^2 = F_k F_{k+1}$ for some natural number k.
   We want to show that $\Sigma_{k=0}^{k+1} {F_k}^2 = F_{k+1} F_{k+1+1}$.
   Begin with the equation $\Sigma_{k=0}^{k} {F_k}^2 = F_k F_{k+1}$, and add an ${F_{k+1}}^2$ to both sides.
   Then you get the equation $\Sigma_{k=0}^{k+1} {F_k}^2 = F_k F_{k+1} + {F_{k+1}}^2$.
   Factoring, we get $\Sigma_{k=0}^{k+1} {F_k}^2 = F_{k+1}(F_k + F_{k+1})$.
   Now use the fact that $F_k + F_{k+1} = F_{k+2}$, and we get $\Sigma_{k=0}^{k+1} {F_k}^2 = F_{k+1} F_{k+2}$.
   This shows $\Sigma_{k=0}^{k+1} {F_k}^2 = F_{k+1} F_{k+1+1}$ so the theorem is proved.