Solutions to Additional Counting Problems

In many cases an unsimplified version of the answer is given in order to indicate what thought process was involved. Ask me if you are unclear about how the answer was obtained.

1. Sections 6 and 7
   (a) 72
   (b) 10000
   (c) $10000 - 10 \cdot 9 \cdot 8 \cdot 7$
   (d) 2^10
   (e) 4^10
   (f) $2 \cdot 26^2 + 2 \cdot 26^3$
   (g) $26^2 10^4$
   (h) $26^2 10 \cdot 9 \cdot 8 \cdot 7$
   (i) $8 \cdot 3 \cdot 2$
   (j) $5! - 8 \cdot 3 \cdot 2$
2. Sections 11 and 12
   (a) i. Recall that a relation on A is a subset of $A \times A$. The number of such subsets is $|2^{A \times A}| = 2^{|A \times A|} = 2^{2 \cdot 2} = 16$. They are listed in the following table. Notice that the null set is listed. It is a subset of $A \times A$, so it is a relation on A. It is the relation where no elements are related.

$\{\}$	$\{(a,b)\}$	$\{(b,a)\}$	$\{(a,b),(b,a)\}$
$\{(a,a)\}$	$\{(a,a),(a,b)\}$	$\{(a,a),(b,a)\}$	$\{(a,a),(a,b),(b,a)\}$
$\{(b,b)\}$	$\{(b,b),(a,b)\}$	$\{(b,b),(b,a)\}$	$\{(b,b),(a,b),(b,a)\}$
$\{(a,a),(b,b)\}$	$\{(a,a),(b,b),(a,b)\}$	$\{(a,a),(b,b),(b,a)\}$	$\{(a,a),(b,b),(a,b),(b,a)\}$

   ii. Those in the last row.
   iii. Those in the first and last columns
   iv. $\{(a,a),(b,b)\}$ and $\{(a,a),(a,b),(b,a),(b,b)\}$
   v. Just those in the second and third columns

   (b) This problem concerns relations on the set $A = \{a, b, c\}$. It is similar to the previous problem except that there are so many relations on A that you will not want to write them all down. Still, you can answer the questions using standard counting arguments.
   i. 2^9
   ii. 2^6
   iii. 2^6
   iv. Hint: How many partitions are there on A? Answer: 5
   v. 2^8

3. SECTION 13
   (a) $\frac{26!}{8!6!9!3!}$
   (b) How many different anagrams of the following words can be made?
      i. $\frac{11!}{3!2!2!2!}$
      ii. $\frac{7!}{2!}$
   (c) $\frac{24!}{4!4!4!4!4!4!}$
   (d) $\frac{9!}{4!3!2!}$
   (e) $\frac{11!}{7!4!}$
   (f) $\frac{8!}{16}$
4. SECTION 14
   (a)
   (b) $\frac{10!}{4!6!}$
   (c) $\frac{12!}{8!4!}\frac{14!}{6!8!}$
   (d) Given a standard 52-card deck,
      i. $\frac{52!}{7!45!}$
      ii. $4\frac{13!}{7!6!}$
      iii. $\frac{13!}{3!10!}3\frac{13!}{4!9!}$
   (e) $\frac{12!}{4!8!}$
   (f) $\frac{12!}{0!12!} + \frac{12!}{1!11!} + \frac{12!}{2!10!} + \frac{12!}{3!9!} + \frac{12!}{4!8!}$
   (g) $x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$.
   (h) $35 \cdot 3^4(-2)^2$
   (i) $\frac{12!}{6!6!}$
   (j) $\frac{12!}{6!6!}$
   (k) $\frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}$
5. SECTION 15
   (a) $\frac{9!}{4!5!}$
   (b) $\frac{27!}{19!8!}$
   (c) $\frac{73!}{3!70!}$
   (d) $\frac{11!}{3!8!}$
   (e) $\frac{10!}{2!8!}$

6. SECTION 16

   (a) $2 \cdot 5^3 + 5^3 + 2^4 - 2^3 - 2^4 - 2 \cdot 5^2 + 2^3 = 325$

   (b)
      i. 155
      ii. 105
      iii. 80
      iv. 35
      v. 45

7. SECTION 20

   (a) Consider the set $A = \{a, b, c, d, e\}$.
      i. 5^5
      ii. $5!$
      iii. $5^5 - 5!$
      iv. 5^4
      v. $3 \cdot 5^4$
      vi. 5^4
      vii. 5^3
      viii. $\frac{5!}{2!3!}4^2$
      ix. $5^4 + 5^4 - 5^3$
      x. $5!$

8. SECTION 21

   (a) You just need to pull out 3 socks and 2 of them will be the same color. This illustrates the pigeonhole principle for the following reason. Call the socks you pulled out Sock 1, Sock 2 and Sock 3. Put them into a set $A = \{$Sock 1, Sock 2, Sock 3 $\}$. Now think of the set $B = \{$ Black, Blue $\}$. Now consider the function $f : A \to B$ defined as $f(x) =$ color of x. Since $|A| = 3 > 2 = |B|$, the pigeonhole principle says f is not injective. This means there are two socks x and y in A for which $f(x) = f(y)$, meaning the socks have the same color.

   (b) Let A be the set of people at the party. Consider the function f whose domain is A and which is defined as $f(x) =$ the number of friends that x has at the party. Notice that $0 \leq f(x) \leq 29$. If there is an x for which $f(x) = 0$, then there is no y for which $f(y) = 29$, for otherwise x would be friends with no one, and y would be friends with everyone, and that's not possible. For the same reason, if there's an y for which $f(y) = 29$, then there is no x for which $f(x) = 0$.

   Thus the function f has one of the following forms:
   $f : A \to \{1, 2, 3, 4, 5, 6, \cdots 28, 29\}$ or
   $f : A \to \{0, 2, 3, 4, 5, 6, \cdots 28\}$.

   In either case, the image of f has fewer that 30 elements. Since $|A| = 30$, the pigeonhole principle says that f is not injective. Therefore, there are elements x and y in A for which $f(x) = f(y)$. In words, x and y have the same number of friends at the party.

   (c) 6