Linear Algebra Test #2 April 28, 2006

Name: ________________ R. Hammack Score: ______

1. Consider the matrix, $A = \begin{bmatrix} 1 & 0 & 3 & 2 \\ 1 & 2 & 7 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 1 & 5 & 1 \end{bmatrix}$.

(a) Find a basis for Col(A).

$$\begin{bmatrix} 1 & 0 & 3 & 2 \\ 1 & 2 & 7 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 1 & 5 & 1 \end{bmatrix} \begin{matrix} R_2 - R_1 \\ R_4 - R_1 \\ \longrightarrow \end{matrix} \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 2 & 4 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & -1 \end{bmatrix} \begin{matrix} R_2 - 2R_3 \\ R_4 - R_3 \\ \longrightarrow \end{matrix} \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 0 & 0 & -3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} \begin{matrix} -\frac{1}{3}R_2 \\ \\ -\frac{1}{2}R_4 \\ \longrightarrow \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 - 2R_2 \\ R_3 - R_2 \\ R_4 - R_2 \\ \longrightarrow \end{matrix} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 \leftrightarrow 2R_3 \\ \longrightarrow \end{matrix} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From this, you can see that it is the first, second and fourth column of A that forms a basis for the column space. **ANSWER:** $\mathscr{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

(b) Find a basis for Null$(A)^{\perp}$.

Since Null$(A)^{\perp}$ = Row(A), it follows from the above work that the answer is
$\mathscr{B} = \{[1\ 0\ 3\ 0], [1\ 0\ 2\ 0], [0\ 0\ 0\ 1]\}$.

2. Suppose linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies $T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ and $T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

Find the standard matrix for T.

Note $T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = T \begin{bmatrix} 1 \\ 1 \end{bmatrix} - T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$.

Then $[T] = [T(\mathbf{e}_1)\ T(\mathbf{e}_2)] = \begin{bmatrix} 2 & 3 \\ -2 & 3 \end{bmatrix}$.

3. Suppose A is a square matrix with eigenvalue λ. Prove that λ^2 is an eigenvalue of A^2

Proof. Suppose A has eigenvalue λ, so there is a vector $\mathbf{x}$ for which $A\mathbf{x} = \lambda\mathbf{x}$. Then:

$AA\mathbf{x} = A(\lambda\mathbf{x})$ (multiply both sides by A)
$A^2\mathbf{x} = \lambda A\mathbf{x}$ (properties of matrix multiplication)
$A^2\mathbf{x} = \lambda\lambda\mathbf{x}$ ($A\mathbf{x} = \lambda\mathbf{x}$)
$A^2\mathbf{x} = \lambda^2\mathbf{x}$ (rewite)

This last equation means λ^2 is an eigenvalue of A^2

4. Consider the orthogonal basis $\mathscr{B} = \left\{ \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$ of $\mathbb{R}^2$. Find $[\mathbf{v}]_{\mathscr{B}}$, if $\mathbf{v} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

$$[\mathbf{v}]_{\mathscr{B}} = \begin{bmatrix} \dfrac{\begin{bmatrix} 2 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \end{bmatrix}}{\begin{bmatrix} 2 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \end{bmatrix}} \\ \\ \dfrac{\begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \end{bmatrix}}{\begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix}} \end{bmatrix} = \begin{bmatrix} -\frac{7}{13} \\ \\ \frac{22}{13} \end{bmatrix}$$

5. This problem concerns the matrix $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$.

   (a) Find the eigenvalues for A.
   $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2$. It follows that the only eigenvalue is $\lambda = 1$.

   (b) Find the eigenspaces for A.
   $E_1 = \text{Null}(A - I) = \text{Null}\left(\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}\right) = \text{Span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

   (c) Is A diagonalizable? Explain.
   No. Parts (a) and (b) above show that $\mathbb{R}^2$ does not have a basis of eigenvectors of A, so A is not diagonalizable.

6. In this problem, A is a 4×4 matrix satisfying $P^{-1}AP = D$,
   where $P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

   (a) List the eigenvalues of A, and for each eigenvalue, give a basis for its eigenspace. (Note: This can be done without computations.)
   The eigenvalues, read off of D are 5, 2 and 1. The eigenspaces, read off of P, are as follows.
   $$E_5 = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right), \qquad E_2 = \text{Span}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}\right), \qquad E_1 = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}\right).$$

   (b) $A \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}$

   (c) Find the determinant of A. Explain your work.
   $P^{-1}AP = D$
   $\det(P^{-1}AP) = \det(D)$
   $\det(P^{-1})\det(A)\det(P) = 20$
   $\dfrac{1}{\det(P)}\det(A)\det(P) = 20$
   $\det(A) = 20$

   (d) Is A invertible? Why or why not?
   Yes, because its determinant is not 0.