Linear Algebra | Test #1 | March 17, 2006

Name: ______________________ R. Hammack Score: ______

1. (30 points) For this problem, $A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 5 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$.

   Perform the indicated operations, if possible. (If it is not possible, say why.)

   (a) $\mathbf{u} - \frac{2}{3}\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -4/3 \\ 2/3 \\ -2 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 5/3 \\ -3 \end{bmatrix}$

   (b) $\mathbf{u} \cdot \mathbf{v} = 2 - 1 - 3 = -2$

   (c) $||\mathbf{u}|| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{1+1+1} = \sqrt{3}$

   (d) $BA = \begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 2 & 3 & -1 \\ 1 & 5 & 4 \end{bmatrix} = \begin{bmatrix} -4 & 1 & 11 \\ 4 & -1 & -4 \end{bmatrix}$

   (e) $AA^T = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 5 & 4 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 5 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 14 & 13 \\ 13 & 42 \end{bmatrix}$

   (f) $B^{-1} =$ (Matrix B is not invertible because $ad - bc = 0$.)

2. (10 points) Suppose C, D and X are invertible matrices. Solve the equation $C(D + X^{-1}) = DC + CD$ for X. Simplify your answer as much as possible.

   $C(D + X^{-1}) = DC + CD$
   $CD + CX^{-1} = DC + CD$
   $CX^{-1} = DC + CD - CD$
   $CX^{-1} = DC$
   $C^{-1}CX^{-1} = C^{-1}DC$
   $X^{-1} = C^{-1}DC$
   $(X^{-1})^{-1} = (C^{-1}DC)^{-1}$
   $X = C^{-1}D^{-1}(C^{-1})^{-1}$
   $X = C^{-1}D^{-1}C$

3. (15 points) Solve the following system.

$$\begin{array}{rcrcrcl} 2x & + & y & + & 6z & = & 0 \\ x & + & 3y & + & 3z & = & 0 \\ x & - & 2y & + & 3z & = & 0 \\ 2x & + & 6y & + & 6z & = & 0 \end{array}$$

Setting up the augmented matrix and row-reducing, we get:

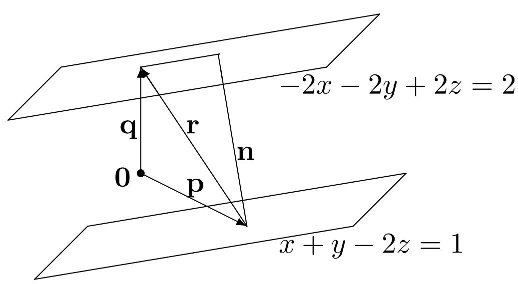
$$\left[\begin{array}{rrr|r} 2 & 1 & 6 & 0 \\ 1 & 3 & 3 & 0 \\ 1 & -2 & 3 & 0 \\ 2 & 6 & 6 & 0 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_1} \left[\begin{array}{rrr|r} 1 & 3 & 3 & 0 \\ 2 & 1 & 6 & 0 \\ 1 & -2 & 3 & 0 \\ 2 & 6 & 6 & 0 \end{array}\right] \xrightarrow{\substack{R_2 - 2R_1 \\ R_3 - R_1 \\ R_4 - 2R_1}} \left[\begin{array}{rrr|r} 1 & 3 & 3 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 - R_2} \left[\begin{array}{rrr|r} 1 & 3 & 3 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-\frac{1}{5}R_2} \left[\begin{array}{rrr|r} 1 & 3 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 - 3R_2} \left[\begin{array}{rrr|r} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The row-reduced matrix corresponds to the equivalent system $\left\{ \begin{array}{rcrcl} x & & + & 3z & = 0 \\ & y & & & = 0 \end{array} \right.$

The solutions of this system are $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = z \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$, $z \in \mathbb{R}$.

4. (15 points) Find the distance between the parallel planes $x + y - 2z = 1$ and $-2x - 2y + 4z = 2$ in $\mathbb{R}^3$. Please explain your reasoning. Illustrate it with a picture if necessary.



The tip of $\mathbf{p} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is on the first plane and the tip of $\mathbf{q} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$ is on the second plane, as illustrated. The vector $\mathbf{n} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ is normal to both planes. From the diagram, the distance between the planes is the length of the projection of $\mathbf{r} = \mathbf{q} - \mathbf{p}$ $= \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$ onto $\mathbf{n}$. That is, the distance is $\|\text{proj}_{\mathbf{n}}(\mathbf{q} - \mathbf{p})\| = \left\| \frac{\mathbf{r} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}} \mathbf{n} \right\| = \left\| \frac{-2}{6} \mathbf{n} \right\| = \frac{1}{3} \|\mathbf{n}\| = \frac{\sqrt{6}}{3}$.

5. (15 points) Is the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ invertible? If so, find its inverse. If not, say why.

Using the Gauss-Jordan method,

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_3 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -2 & -1 & 1 & -2 \end{array}\right] \xrightarrow{-\frac{1}{2}R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1/2 & -1/2 & 1 \end{array}\right] \xrightarrow[R_2 - R_3]{R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & -1 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 1/2 & -1/2 & 1 \end{array}\right]$$

Therefore the matrix is invertible, and its inverse is $\begin{bmatrix} 1/2 & 1/2 & -1 \\ -1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 1 \end{bmatrix}$

6. (15 points) This problem concerns the three vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in $\mathbb{R}^3$.

(a) Are these vectors linearly independent or linearly dependent? Explain. (Suggestion: Compare these vectors to the matrix in Problem 5.)

To answer this question, it is necessary to investigate the solutions of the equation

$$x \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + z \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

In matrix form, this is $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. However, the matrix on the left is invertible, by Problem 5. Multiplying both sides of the above equation by the inverse of the matrix gives a unique solution $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Since we get only the trivial solution, the vectors are **linearly independent**.

(b) Is $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ in the span of these vectors? Explain.

Yes. Note $1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, so the vector in question is a linear combination of the three given vectors, so it is in their span.