Richard Hammack's
MATH 756

Chapter 8

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Goal Operator $S : \Gamma_0 \to \Gamma$ for which

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Given a graph G, graph $S(G)$ will be called its **Cartesian skeleton**.

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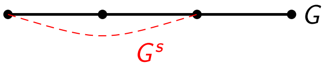

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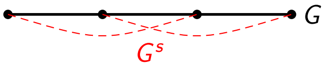

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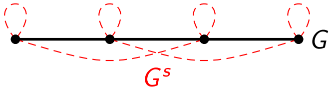

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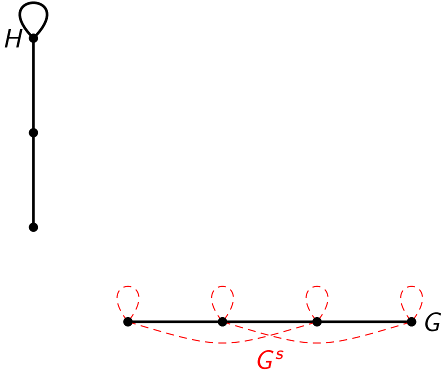

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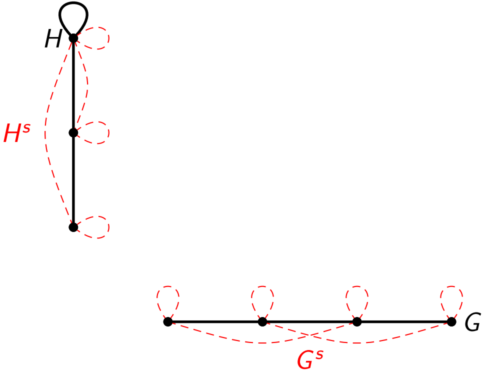

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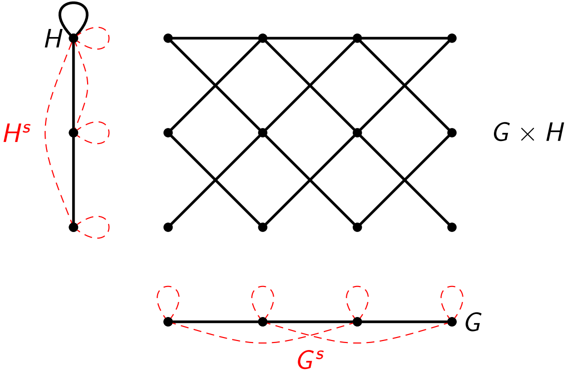

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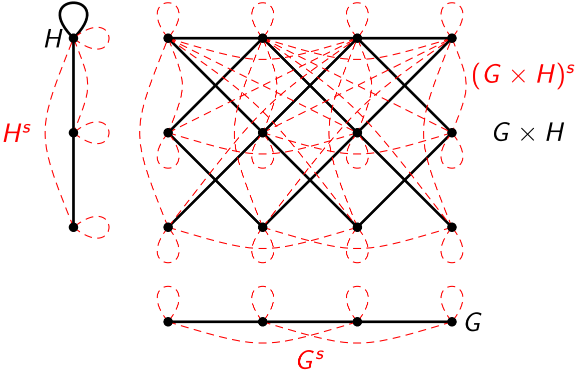

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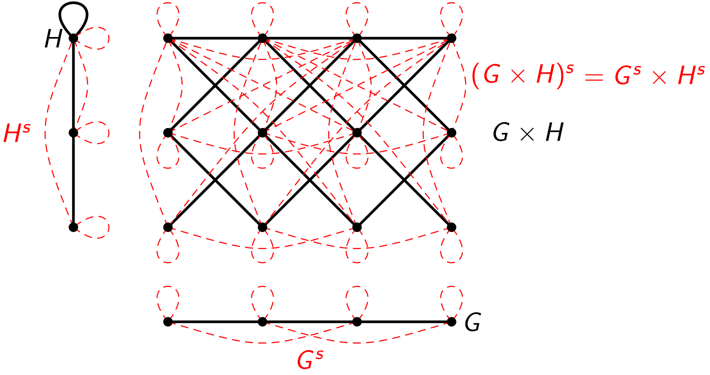

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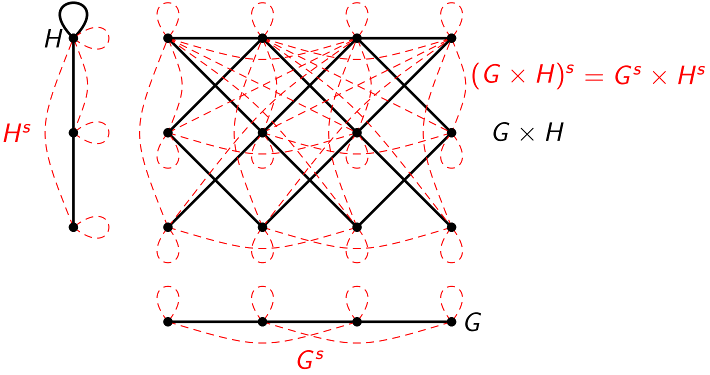

Lemma 8.8 $(G_1 \times \cdots \times G_k)^s = G_1^s \times \cdots \times G_k^s$.

The Cartesian Skeleton

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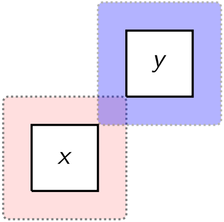
Motivation: Suppose a wall (graph) is made of bricks (vertices).
$N(x)$ denotes mortar around brick x.
How can you tell when adjacent bricks x & y are at a diagonal?



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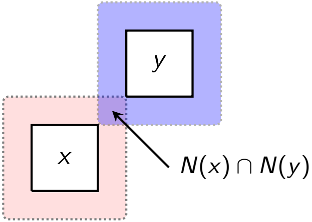

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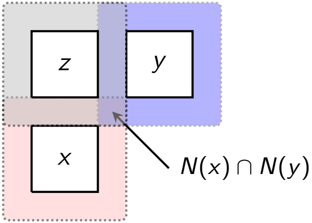

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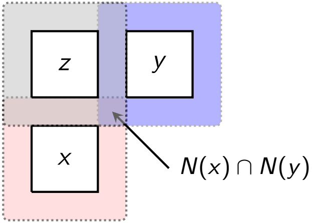

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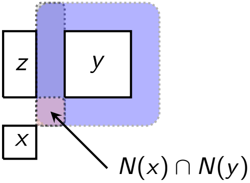

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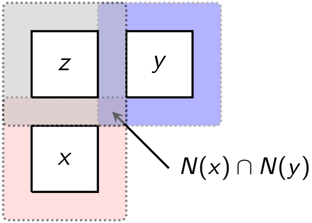

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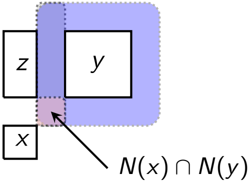

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This is equivalent to **both** of the following holding.

1. $N(x) \cap N(y) \subset N(x) \cap N(z)$ or $N(x) \subset N(z) \subset N(y)$
2. $N(y) \cap N(x) \subset N(y) \cap N(z)$ or $N(y) \subset N(z) \subset N(x)$.

The Cartesian Skeleton

Edge xy of G^s is **dispensable** if $x = y$ or $\exists\, z \in V(G)$ for which both:

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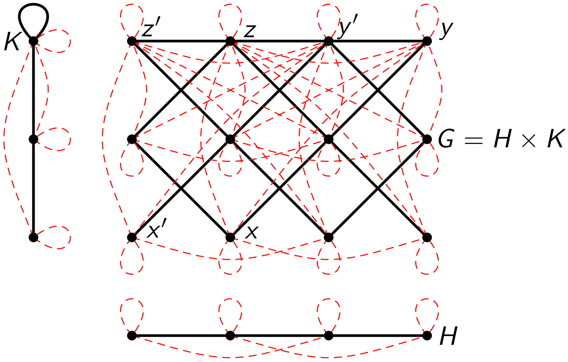

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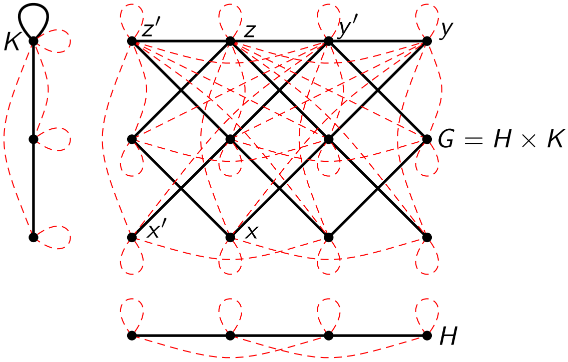

Examples: Loops dispensable; xy dispensable; $x'y'$ dispensable.

The Cartesian Skeleton

Cartesian Skeleton of G is graph $S(G)$ with:

$$\begin{aligned} V(\,S(G)\,) &= V(G) \\ E(\,S(G)\,) &= \{\,xy \in G^s \;\mid\; xy \textbf{ is NOT dispensable}\} \end{aligned}$$

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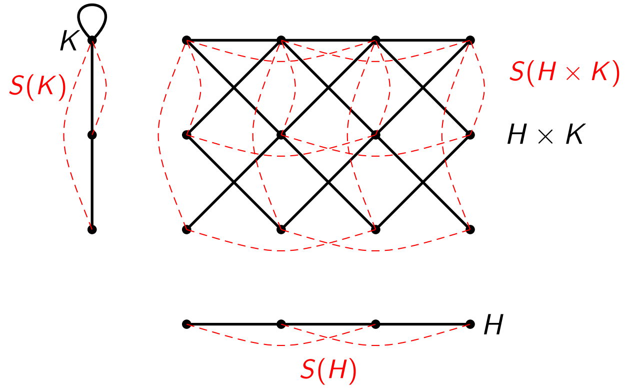

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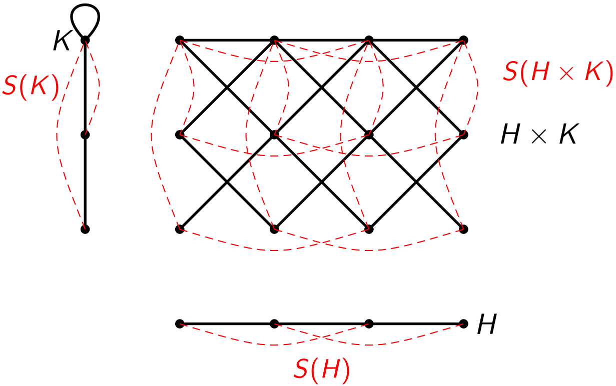

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Proposition 8.10: If H, K are R-thin, then $S(H \times K) = S(H) \Box S(K)$.

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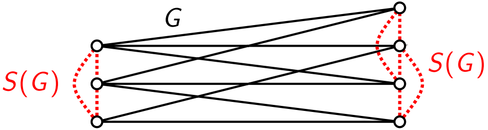
- If G bipartite, $S(G)$ has exactly two components; their respective vertex sets are the two partite sets of G.



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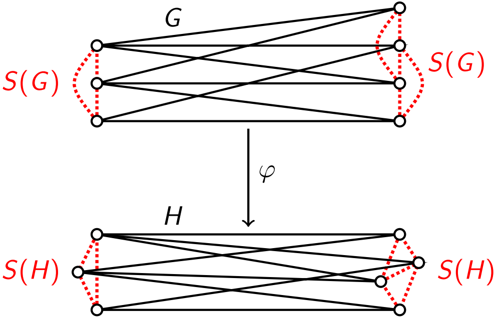

Proposition 8.11: Any isomorphism $\varphi : G \to H$ is also an isomorphism $\varphi : S(G) \to S(H)$.

Our Plan

- §**8.4 Factoring Connected Nonbipartite R-thin Graphs.**
  Use $S(G_1 \times \cdots \times G_k) = S(G_1)\Box \cdots \Box S(G_k)$ to get:

  Theorem. Connected nonbipartite R-thin graphs in Γ_0 factor uniquely into primes (w.r.t. $\times$)

- §**8.5 Factoring Connected Nonbipartite Graphs.**
  Remove restriction to R-thin

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