MATH 756
About graph exponentiation

Richard Hammack

RH & Cristy Mulligan, (2017) Neighborhood reconstruction & graph cancellation, *Electronic Journal of Combinatorics*, **24**(2).

RH, (2021) Graph exponentiation and neighborhood reconstruction, *Discussiones Mathematicae Graph Theory*, 41, 335–339.

Graph exponentiation G^K

$$G^K = \begin{cases} V(G^K) & = & \text{Set of all functions } f : V(K) \to V(G) \\ E(G^K) & = & \{fg \mid f(x)g(y) \in E(G) \ \forall\, xy \in E(K)\,\} \end{cases}$$

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If $V(K) = \{v_1, \, v_2, \, \ldots, \, v_{|K|}\}$, then $f : V(K) \to V(G)$ is denoted
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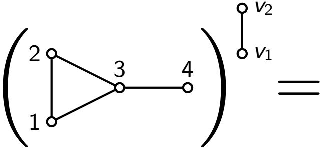
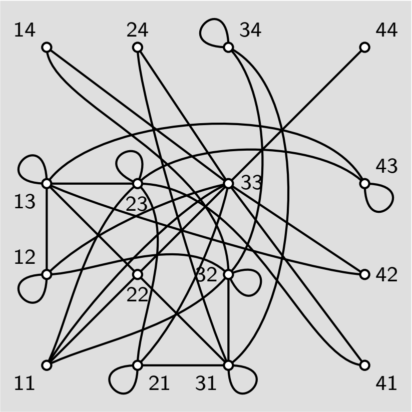
Example





Properties of exponentiation

Notation: $\mathbb{I} = (\{a\}, \{aa\}) =$ 

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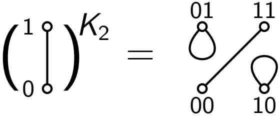

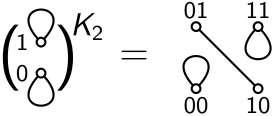

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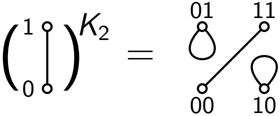

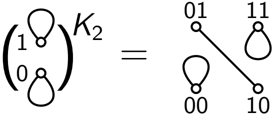

Question: For which G does $G^K \cong H^K \Rightarrow G \cong H$?

Question (scaled down)

For which G does $G^{K_2} \cong H^{K_2} \Rightarrow G \cong H$?

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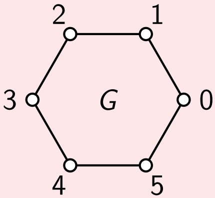

$$\mathscr{N}(G) = \{\{0,2\},\{2,4\},\{0,4\},\{1,3\},\{3,5\},\{1,5\}\}$$

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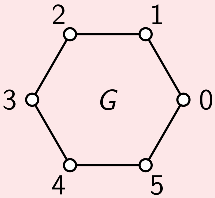

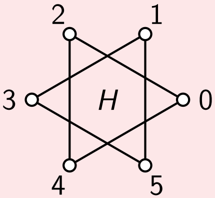

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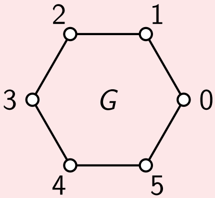

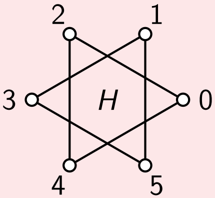

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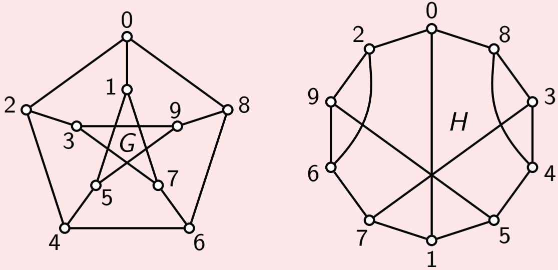
G **neighborhood reconstructible** if $\mathscr{N}(G) = \mathscr{N}(H) \implies G \cong H$

Petersen graph NOT neighborhood reconstructible



$$
\begin{array}{lllll}
N_G(0) & = & \{1,2,8\} & = & N_H(0) \\
N_G(1) & = & \{0,5,7\} & = & N_H(1) \\
N_G(2) & = & \{0,3,4\} & = & N_H(8) \\
N_G(3) & = & \{2,7,9\} & = & N_H(6) \\
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Theorem. Graph G is neighborhood reconstructible if and only if $G^{K_2} \cong H^{K_2} \;\Rightarrow\; G \cong H$ for all graphs H.

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Therefore $G^{K_2} \cong H^{K_2}$, so $G \cong H$ by $\heartsuit$. $\square$

Open questions (some low hanging fruit?)

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Thank You!