Test Two — Advanced Linear Algebra, MATH 610, R. Hammack — November 1, 2018

Name: KEY

Score: ______

Directions: All spaces are finite dimensional. Each question (including parts) is 10 points.

1. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

(a) Find the characteristic polynomial χ_A of A. (Show your work.)

$$\chi_A = |xI - A| = \begin{vmatrix} x-1 & 0 & 0 \\ 0 & x-1 & 0 \\ 0 & 1 & x-1 \end{vmatrix} = \boxed{(x-1)^3}$$

(b) Find the minimum polynomial m_A of A. (Explain your ressoning.)

We know m_A and χ_A must have the same roots, so $m_T = (x-1)^k$ for some k.

Certainly $m_T \neq (x-1)$, because $(A-I) \neq 0$

But notice that $(A-I)^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Therefore $\boxed{m_A = (x-1)^2}$

(c) Is A diagonalizable? (Explain.)

NO We know A is diagonalizable if and only if m_A factors into distinct linear factors, and that is not the case here.

2. Give an example of an operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ for which $\mathbb{R}^2 = \text{Range}(T) \oplus \text{Null}(T)$ but T is not a projection.

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be defined as $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$,

that is, $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 0 \end{bmatrix}$

Then Range(T) is the x-axis of $\mathbb{R}^2$.
Also Null(T) is the y-axis of $\mathbb{R}^2$

Therefore $\mathbb{R}^2 = \text{Range}(T) \oplus \text{Null}(T)$.

However, T is not a projection because

$$T^2\begin{bmatrix} x \\ y \end{bmatrix} = TT\begin{bmatrix} x \\ y \end{bmatrix} = T\begin{bmatrix} 2x \\ 0 \end{bmatrix} = \begin{bmatrix} 4x \\ 0 \end{bmatrix} = 2\begin{bmatrix} 2x \\ 0 \end{bmatrix} = 2T\begin{bmatrix} x \\ y \end{bmatrix}.$$

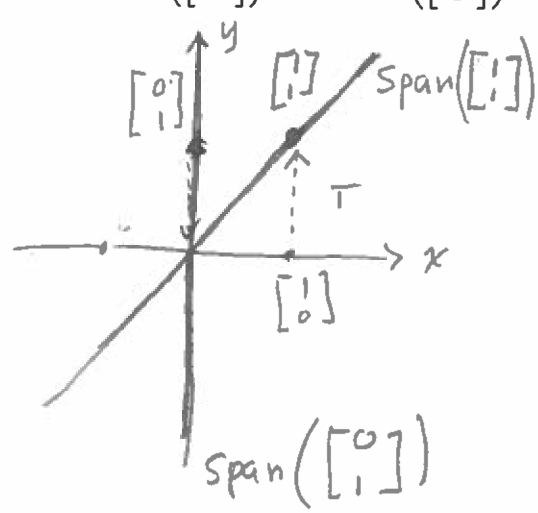
Hence $T^2 = 2T \neq T$ (so T is not a projection)

3. Find the matrix (relative to the standard basis) for the projection that projects $\mathbb{R}^2$ to Span$\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ along Span$\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$.

Denote this projection as T.

Notice that: $T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Therefore

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

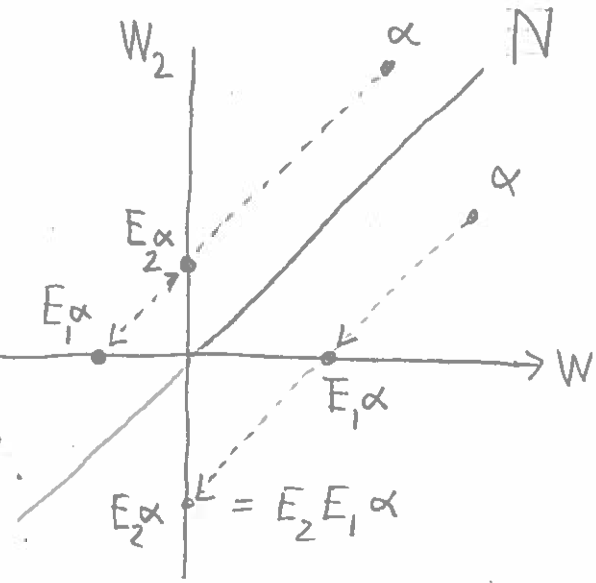


Answer:
Matrix for T is $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

4. Suppose $V = W_1 \oplus W_2$. Let E_1 be a projection to W_1. Let E_2 be a projection to W_2.

(a) Is it necessarily true that $E_1E_2 = O$? Explain.

<u>No</u> Here is a counterexample.
Let W_1 be The x-axis of $\mathbb{R}^2$
Let W_2 be The y-axis of $\mathbb{R}^2$
Let N be The line $y = x$
Let E_1 be projection to W_1 along N
Let E_2 be projection to w_2 along N.



Notice that from the drawing we get $\boxed{E_2E_1 = E_2}$ and $\boxed{E_1E_2 = E_1}$

Therefore $E_1E_2 = E_1 \neq O$.

(b) Is it necessarily true that $E_1 + E_2$ is a projection? Explain.

Using the same E_1 and E_2 as above, as well as the boxed equations, we get

$$
\begin{aligned}
(E_1+E_2)^2 &= (E_1+E_2)(E_1+E_2) \\
&= E_1^2 + E_1E_2 + E_2E_1 + E_2^2 \\
&= E_1 + E_1 + E_2 + E_2 \\
&= 2E_1 + 2E_2 \\
&= 2(E_1+E_2)
\end{aligned}
$$

Thus $(E_1+E_2)^2 = 2(E_1+E_2) \neq E_1+E_2$

so $\boxed{E_1+E_2 \text{ is } \underline{\text{not}} \text{ a projection}}$

5. Suppose $E, T \in L(V, V)$ and E is a projection onto the subspace $W \subseteq V$.
Prove that W is T-invariant if and only if $ETE = TE$.

Proof

($\Rightarrow$) Suppose W is T-invariant.

Take any $\alpha \in V$ and notice that $E\alpha \in W$, and hence $TE\alpha \in W$ (because W is T-invariant).

But we know that for any $\beta \in W$, $E\beta = \beta$.

Therefore from $TE\alpha \in W$ we get

$$E(TE\alpha) = TE\alpha.$$

Now we've established $ETE\alpha = TE\alpha$ for all $\alpha \in V$, which means $ETE = TE$.

($\Leftarrow$) Suppose $ETE = TE$,

We need to show W is T-invariant

So take any $\alpha \in W$. We must show $T\alpha \in W$.

Because $\alpha \in W$ we know that $\alpha = E\alpha$

$$\begin{aligned} \text{Therefore } T\alpha &= TE\alpha \\ &= ETE\alpha \quad (\text{because } ETE = E) \\ &= E(TE\alpha) \in W. \quad (\text{because } E \text{ is projection to } W) \end{aligned}$$

Thus we thave $T\alpha \in W$ so W is T-invariant.

6. Suppose $T \in L(V, V)$ and every subspace of V is T-invariant. Prove that T is a scalar multiple of the identity.

Proof Select a basis $\mathcal{B} = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ for V. Then each subspace $\mathrm{Span}(\alpha_i)$ is a one-dimensional subspace of V, and it is also T-invariant by assumption.

Consequently $T\alpha_i$ can only be a scalar multiple of α_i, so $T\alpha_i = c_i \alpha_i$ for some scalar c_i.

Thus we have scalars c_1 c_2 ... c_n for which $T\alpha_1 = c_1\alpha_1$, $T\alpha_2 = c_2\alpha_2$, ... $T\alpha_n = c_n\alpha_n$.

Now choose any index $i \neq 1$ and consider the 1-dimensional T-invariant subspace $\mathrm{Span}(\alpha_1 + \alpha_i)$. Note $T(\alpha_1 + \alpha_i)$ can only be a scalar multiple of $\alpha_1 + \alpha_i$, so $T(\alpha_1 + \alpha_i) = \lambda(\alpha_1 + \alpha_i)$ for some scalar λ. Now we have

$$\underline{c_1\alpha_1 + c_i\alpha_i} = T\alpha_1 + T\alpha_i = T(\alpha_1 + \alpha_i) = \lambda(\alpha_1 + \alpha_i) = \underline{\lambda\alpha_1 + \lambda\alpha_i}$$

From this we obtain $(c_1 - \lambda)\alpha_1 + (c_i - \lambda)\alpha_i = 0$, and since α_1 and α_i are independent it follows that $c_1 - \lambda = 0$ and $c_i - \lambda = 0$. Hence $c_1 = \lambda = c_i$. Since this is true for any i, we have $c_1 = c_2 = c_3 = \cdots c_n = \lambda$. So $T\alpha_i = c\alpha_i$ $\forall i$

Thus $\boxed{T = cI}$

7. Let V be the vector space of $n \times n$ matrices over a field $\mathbb{F}$ and let $A \in V$ be a fixed matrix. Define a linear operator $T: V \to V$ as $T(X) = AX$. Show that the minimum polynomial of T equals the minimum polynomial of A.

Notice that
$$TX = AX$$
$$T^2X = TTX = TAX = A^2X$$
$$T^3X = TT^2X = TA^2X = A^3X.$$

In general, $T^kX = A^kX$.

More generally if $f \in \mathbb{F}[x]$, then

$$\boxed{f(T)X = f(A)X} \quad *$$

Let m_T be the minimal poly of T and let m_A be the minimal poly of A.

Notice that $m_A(T)X \underset{\text{by } *}{=} m_A(A)X \underset{\text{def. of } m_A}{=} 0X = 0$

Since $m_A(T)X = 0$ for all X we get $\boxed{m_A(T) = 0}$

Consequently m_A is in the ideal $I = \{f \in \mathbb{F}[x] \mid f(T) = 0\}$ which is generated by m_T. From this we conclude $\boxed{m_T \text{ divides } m_A}$

Next observe $m_T(A)X \underset{\text{by } *}{=} m_T(T)X \underset{\text{def of } m_T}{=} 0X = 0$

As before we conclude $m_T(A) = 0$, so m_T is in the ideal $I = \{f \in \mathbb{F}[x] \mid f(A) = 0\}$ which is generated by m_A. Therefore $\boxed{m_A \text{ divides } m_T}$

Since m_T and m_A are monic polynomials, each dividing the other we get $\boxed{m_A = m_T}$