Test One Advanced Linear Algebra September 27, 2018

MATH 610

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Answer in the space provided. Closed book. No calculators. Please put all phones, etc., away.

1. Suppose $T : V \to W$ is a linear transformation. Prove that the range of T is a subspace of W.

Proof: Suppose $c \in \mathbb{F}$ and $\alpha, \beta \in \text{Range}(T)$. We must show that $c\alpha + \beta \in \text{Range}(T)$.
Because $\alpha, \beta \in \text{Range}(T)$, we know that $\alpha = T(\gamma)$ and $\beta = T(\delta)$ for some $\gamma, \delta \in V$.
Then $c\alpha + \beta = cT(\gamma) + T(\delta) = T(c\gamma) + T(\delta) = T(c\gamma + \delta)$, by linearity of T.
But now we have $c\alpha + \beta = T(c\gamma + \delta) \in \text{Range}(T)$.
It follows that Range(T) is a subspace of W.

2. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation for which $T^2 = T$.

Show that there is a basis $\mathscr{B}$ of $\mathbb{R}^2$ for which $[T]_{\mathscr{B}}$ is one of the matrices $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, or $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Consider the following three mutually exclusive and exhaustive cases.

CASE 1: Say T is the zero transformation, that is, $T(\alpha) = 0$ for all $\alpha \in \mathbb{R}^2$. Let $\mathscr{B} = \{\beta_1, \beta_2\}$ be *any* basis of $\mathbb{R}^2$. Then $T(\beta_1) = 0 = 0\beta_1 + 0\beta_2$, which means $[T(\beta_1)]_{\mathscr{B}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Also $T(\beta_2) = 0 = 0\beta_1 + 0\beta_2$, so $[T(\beta_2)]_{\mathscr{B}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
Now $[T]_{\mathscr{B}} = \big[[T(\beta_1)]_{\mathscr{B}}\ [T(\beta_2)]_{\mathscr{B}}\big] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

CASE 2: Say T is the identity transformation, that is, $T(\alpha) = \alpha$ all $\alpha \in \mathbb{R}^2$. Let $\mathscr{B} = \{\beta_1, \beta_2\}$ be *any* basis of $\mathbb{R}^2$. Then $T(\beta_1) = \beta_1 = 1\beta_1 + 0\beta_2$, which means $[T(\beta_1)]_{\mathscr{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Also $T(\beta_2) = \beta_2 = 0\beta_1 + 1\beta_2$, so $[T(\beta_2)]_{\mathscr{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
Now $[T]_{\mathscr{B}} = \big[[T(\beta_1)]_{\mathscr{B}}\ [T(\beta_2)]_{\mathscr{B}}\big] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

CASE 3: Say T is neither the identity transformation nor the zero transformation. Since $T \neq O$, there is a vector $\gamma \in V$ for which $T(\gamma) \neq 0$. Since $T \neq I$, there is a vector $\delta \in V$ for which $\delta - T(\delta) \neq 0$. Put $\beta_1 = T(\gamma)$ and $\beta_2 = \delta - T(\delta)$. Notice $\boxed{T(\beta_1) = T\big(T(\gamma)\big) = T^2(\gamma) = T(\gamma) = \beta_1}$ and $\boxed{T(\beta_2) = T\big(\delta - T(\delta)\big) = T(\delta) - T^2(\delta) = T(\delta) - T(\delta) = 0}$.

Put $\mathscr{B} = \{\beta_1, \beta_2\}$, which is a basis as follows: As $|\mathscr{B}| = 2 = \dim(\mathbb{R}^2)$ we only need to verify that $\mathscr{B}$ is independent. If $x\beta_1 + y\beta_2 = 0$, then $T(x\beta_1 + y\beta_2) = T(0)$, which is $xT(\beta_1) + yT(\beta_2) = 0$, or $x\beta_1 + 0 = 0$, and this implies $x = 0$. Then from $x\beta_1 + y\beta_2 = 0$ we get $y\beta_2 = 0$, so $y = 0$ too. Thus the set $\mathscr{B}$ is linearly independent, and hence a basis.

Note $T(\beta_!) = \beta_1 = 1\beta_1 + 0\beta_2$, so $[T(\beta_1)]_{\mathscr{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Also $T(\beta_2) = 0 = 0T(\beta_1) + 0\beta_2$, so $[T(\beta_2)]_{\mathscr{B}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
Thus $[T]_{\mathscr{B}} = \Big[\big[T(\beta_1)\big]_{\mathscr{B}}\ \ [T(\beta_2)]_{\mathscr{B}}\Big] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

3. Suppose $T : V \to V$ is a linear operator on a 3-dimensional vector space V.

Suppose there is a vector $\alpha \in V$ for which $T^2(\alpha) \neq 0$ but $T^3(\alpha) = 0$.

(a) Show that the set $\mathscr{B} = \{\alpha,\ T(\alpha),\ T^2(\alpha)\}$ is a basis for V.

Because $|\mathscr{B}| = 3 = \dim(V)$, we only need to verify that $\mathscr{B}$ is independent. Thus suppose

$$x\alpha + yT(\alpha) + zT^2(\alpha) = 0. \tag{1}$$

Apply T to both sides. We get $T\big(x\alpha + yT(\alpha) + zT^2(\alpha)\big) = T(0)$, which is $xT(\alpha) + yT^2(\alpha) + zT^3(\alpha) = 0$, or

$$xT(\alpha) + yT^2(\alpha) = 0 \tag{2}$$

(since $T^3(\alpha) = 0$). From (2) we get $T\big(xT(\alpha) + yT^2(\alpha)\big) = T(0)$, which is $xT^2(\alpha) + yT^3(\alpha) = 0$, or

$$xT^2(\alpha) = 0. \tag{3}$$

From (3) we get $x = 0$. Plugging this into (2) yields $y = 0$. Then (1) yields $z = 0$. Since $x = y = z = 0$, we have shown that $\mathscr{B}$ is independent, hence a basis.

(b) Find the matrix of T relative to $\mathscr{B}$, that is, find $[T]_{\mathscr{B}}$.

Note $T(\alpha) = 0\alpha + 1T(\alpha) + 0T^2(\alpha)$, which means $[T(\alpha)]_{\mathscr{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Also $T\big(T(\alpha)\big) = 0\alpha + 0T(\alpha) + 1T^2(\alpha)$, which means $\big[T\big(T(\alpha)\big)\big]_{\mathscr{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Finally $T\big(T^2(\alpha)\big) = 0 = 0\alpha + 0T(\alpha) + 0T^2(\alpha)$, which means $\big[T\big(T^2(\alpha)\big)\big]_{\mathscr{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Consequently $[T]_{\mathscr{B}} = \Big[\big[T(\alpha)\big]_{\mathscr{B}} \ \ \big[T\big(T(\alpha)\big)\big]_{\mathscr{B}} \ \ \big[T\big(T^2(\alpha)\big)\big]_{\mathscr{B}}\Big] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

4. Consider the basis $\mathscr{B} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ of $\mathbb{R}^2$. Find the dual basis $\mathscr{B}^*$.

Let's begin by finding two functionals that are zero on the second and first basis element, respectively.

Define $f_1 \in (\mathbb{R}^2)^*$ as $f_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x$. Then $f_1\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = 1$ and $f_1\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 0$.

Define $f_2 \in (\mathbb{R}^2)^*$ as $f_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x + y$. Then $f_2\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = 0$ and $f_2\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 1$.

Then $\mathscr{B}^* = \{f_1, f_2\}$.

5. Suppose a vector space V has basis $\mathscr{B} = \{\beta_1, \beta_2, \ldots, \beta_n\}$ and dual basis $\mathscr{B}^* = \{f_1, f_2, \ldots, f_n\}$.

Let $\alpha \in V$. Derive the formula $\alpha = \sum_{i=1}^{n} f_i(\alpha)\beta_i$.

Take an arbitrary $\alpha \in V$. We know we can write α as $\alpha = \sum_{j=1}^{n} c_j\beta_j$. Now observe that

$$\begin{aligned}
\sum_{i=1}^{n} f_i(\alpha)\beta_i &= \sum_{i=1}^{n} f_i\left(\sum_{j=1}^{n} c_j\beta_j\right)\beta_i \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} f_i\left(c_j\beta_j\right)\beta_i \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} c_j f_i\left(\beta_j\right)\beta_i \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} c_j\delta_{ij}\beta_i \\
&= \sum_{i=1}^{n} c_i\beta_i \quad = \quad \alpha
\end{aligned}$$

6. State the definition of the transpose T^t of a linear transformation $T : V \to W$.

The transpose T^t is the linear transformation $T^t : W^* \to V^*$ defined as $T^t(f) = fT$.

7. Suppose V is the space of all polynomials with coefficients in $\mathbb{R}$, and let $D : V \to V$ be the differentiation operator. (That is, $D(f)$ is the derivative of f.)

(a) Describe the null space of D^t.

We claim that the null space of D^t is trivial, that is, $\text{Null}(D^t) = \{0\}$, where 0 is the zero functional $0 \in V^*$. To see this, suppose $f \in \text{Null}(D^t)$. We will show that $f = 0$. Because $f \in \text{Null}(D^t)$, we know $D^t(f) = 0$, which by definition of the transpose means $fD = 0$. Therefore

$$\boxed{f(D(g)) = 0 \text{ for any polynomial } g.}$$

Now let $p \in V$ be any polynomial, and choose any antiderivative $P = \int p(x)dx$. That is, P is a polynomial for which $D(P) = p$. Notice that $f(p) = f(D(P)) = 0$ (by the above boxed equation).

In summary, our functional $f \in \text{Null}(D^t)$ has the property $f(p) = 0$ for *any* polynomial $p \in V$. Thus f is the zero functional. Consequently $\text{Null}(D^t) = \{0\}$.

(b) Let $f \in V^*$ be defined as $f(p) = \int_0^1 p(x)dx$. Find $D^t(f)$. That is, for any $p \in V$, give a formula for $D^t(f)(p)$.

Answer: $D^t(f)(p) = fD(p) = f(p') = \int_0^1 p'(x)dx = p(1) - p(0)$, by the Fundamental Theorem of Calculus.

Thus $\boxed{D^t(f)(p) = p(1) - p(0).}$