Quiz 3 | Advanced Linear Algebra | November 20, 2018

MATH 610

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Score: 10

**Directions:** There are TWO pages. Please answer in the space provided. No calculators. Please put all phones, etc., away.

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1. State the Primary Decomposition Theorem.

Suppose $T$ is a linear operator on a finite-dimensional space $V$. Suppose also that the minimal polynomial $m_T$ of $T$ has prime factorization $m_T = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$. Let $W_i$ be the null space of $p_i(T)^{r_i}$ for each $1 \leq i \leq k$. Then:

(i) $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$

(ii) Each $W_i$ is $T$-invariant

(iii) The restriction of $T$ to $W_i$ has minimum polynomial $p_i^{r_i}$.

2. State the Cyclic Decomposition Theorem.

Suppose $T$ is a linear operator on a finite-dimensional space $V$. Then there exist non-zero vectors $\alpha_1\ \alpha_2 \ldots \alpha_k$ with corresponding $T$-annihilators $p_1\ p_2 \cdots p_k$ such that

(i) $V = Z(\alpha_1, T) \oplus Z(\alpha_2, T) \oplus \cdots \oplus Z(\alpha_k, T)$

(ii) $p_1$ is the minimimium polynomial of $T$

(iii) $p_i$ divides $p_{i-1}$ for each $2 \leq i \leq k$

Furthermore the $p_i$ and $k$ are uniquely determined by (i) & (iii).

3. Suppose $A$ is a $5 \times 5$ matrix with complex entries and minimum polynomial $m_A = (x-4)(x-1)^2$. Given that this is all you know about $A$, list its possible Jordan forms.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

4. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the transformation defined as $T\alpha = A\alpha$, where $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \\ 0 & 4 & 1 \end{bmatrix}$.

The minimum polynomial of $T$ happens to be $m_T = x^3 - 4x^2 - x + 4$.

(a) Say $\alpha_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Find a basis for $Z(\alpha_1; T)$.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{A} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \xrightarrow{A} \begin{bmatrix} 3 \\ 6 \\ 8 \end{bmatrix}$$

Notice That the vectors $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, A\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, A^2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 8 \end{bmatrix} \right\}$ are linearly independent and Thus form a basis for $\mathbb{R}^3$. Hence a basis for $Z(\alpha_1, T)$ is

$$\mathcal{B} = \{ \alpha_1, A\alpha_1, A^2\alpha_1 \} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 8 \end{bmatrix} \right\}$$

(b) Using your answer from part (a), find the rational form of $T$.

Relative to the above basis $\mathcal{B}$ the matrix for $T$ is the companion matrix for the minimum polynomial of $T$, namely

$$\begin{bmatrix} 0 & 0 & -4 \\ 1 & 0 & 1 \\ 0 & 1 & 4 \end{bmatrix}$$