Quiz 2 | Advanced Linear Algebra | October 11, 2018

MATH 610

Name: ________________ R. Hammack Score: ______

Directions: There are TWO pages. Please answer in the space provided. No calculators. Please put all phones, etc., away.

1. Let $A = \begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix}$. Find the monic polynomial $p \in \mathbb{R}[x]$ of lowest degree for which $p(A) = O$.

   That is, find the monic generator of the ideal $I = \{f \in \mathbb{R}[x] \mid f(A) = O\}$.

   Solution: Note that p must have degree greater than 1 because otherwise $p(x) = x + a$ for some $a \in \mathbb{R}$, and then $p(A) = \begin{bmatrix} 2+a & 5 \\ 0 & 1+a \end{bmatrix}$, which does not equal O for any a.

   Consider $p(x) = (x-2)(x-1) = x^2 - 3x + 2$, which is monic of degree 2.

   Observe that $p(A) = (A - 2I)(A - I) = \begin{bmatrix} 0 & 5 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$.

   ANSWER: $p(x) = x^2 - 3x + 2$.

2. Let $V = \{(x_1, x_2, x_3, x_4, x_5, \ldots) \mid x_i \in \mathbb{R}\}$ be the vector space (over $\mathbb{R}$) of all infinite sequences with terms in $\mathbb{R}$. Let $T : V \to V$ be defined as $T(x_1, x_2, x_3, x_4, x_5, \ldots) = (x_2, x_3, x_4, x_5, x_6, \ldots)$. That is, T shifts each term of an input sequence one position to the left, dropping the first term. Example: $T(1, 1, 2, 3, 5, 8, 13, 21, \ldots) = (1, 2, 3, 5, 8, 13, 21, \ldots)$.

   Describe the eigenvectors of T.

   Solution: Suppose $(x_1, x_2, x_3, x_4, x_5, \ldots)$ is an eigenvector with eigenvalue λ.

   Then $T(x_1, x_2, x_3, x_4, x_5, \ldots) = \lambda(x_1, x_2, x_3, x_4, x_5, \ldots)$.

   Therefore $(x_2, x_3, x_4, x_5, x_6, \ldots) = (\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4, \lambda x_5, \ldots)$.

   Consequently $x_2 = \lambda x_1$, $x_3 = \lambda x_2$, $x_4 = \lambda x_3$, etc., and in general $x_{n+1} = \lambda x_n$.

   This means $(x_1, x_2, x_3, x_4, x_5, \ldots)$ is a geometric sequence in which any term is λ times the previous term.

   That is, any eigenvector has form $(x_1, \lambda x_1, \lambda^2 x_1, \lambda^3 x_1, \lambda^4 x_1, \lambda^5 x_1, \ldots)$ for $x_1 \neq 0$ and $\lambda \in \mathbb{R}$.

   ANSWER: The eigenvectors of T are precisely the geometric sequences. (And for any such geometric sequence, its associated eigenvalue is its common ratio.)

3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation defined as $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

(a) Find the eigenvalues of T.

$$\chi_T(x) = \left|A - xI\right| = \begin{vmatrix} 3-x & 4 \\ -1 & -1-x \end{vmatrix} = (3-x)(-1-x) + 4 = x^2 - 2x + 1 = (x-1)^2.$$

Therefore the only eigenvalue is 1.

(b) Find the eigenspaces of T.

The eigenspace for 1 is the null space of $A - 1I = A - I = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}$, that is,

the eigenspace for 1 is the set of all vectors $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ satisfying $\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Solving this system with row reduction yields $\left[\begin{array}{cc|c} 2 & 4 & 0 \\ -1 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$, or $x = -2y$.

Therefore the eigenspace for 1 is the subspace $\left\{\begin{bmatrix} -2y \\ y \end{bmatrix} \mid y \in \mathbb{R}\right\} = \text{Span}\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$.

This is the only eigenspace, and it is one-dimensional.

(c) Is T diagonalizable? Explain.

No. There is only one eigenspace, and it is one-dimensional.

Thus the sum of the dimensions of the eigenspaces does not equal the dimension of the whole space $\mathbb{R}^2$.

It is therefore impossible to find a basis for the two-dimensional space $\mathbb{R}^2$ that consists of eigenvectors for T.

Consequently T is NOT diagonalizable.