Quiz 1 | Advanced Linear Algebra | September 6, 2018
Name: Richard | MATH 610 | Score: 50
R. Hammack

Directions: There are TWO pages. Please answer in the space provided. No calculators. Please put all phones, etc., away.

1. State what it means for a subset S of a vector space V over $\mathbb{F}$ to be **linearly dependent**.

There is a finite subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq V$ and scalars $x_1\ x_2 \cdots x_n \in \mathbb{F}$, not all zero for which $x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n = \vec{0}$.

2. Let V be the vector space (over $\mathbb{R}$) of all functions $f : \mathbb{R} \to \mathbb{R}$.
Let $W = \{f \in V \mid f(-x) = f(x) \text{ for all } x \in \mathbb{R}\}$. That is, W is the set of all *even* functions in V.
Let $X = \{f \in V \mid f(-x) = -f(x) \text{ for all } x \in \mathbb{R}\}$. That is, X is the set of all *odd* functions in V.

(a) Prove that W is a subspace of V. (Note that X is a also a subspace of V, but you don't need to prove it.)

Suppose $c \in \mathbb{F}$ and $f, g \in W$. We must show $cf + g \in W$. Now, because $f, g \in W$ we know $f(-x) = f(x)$ and $g(-x) = g(x)$ for all $x \in \mathbb{R}$.
Note $(cf+g)(-x) = cf(-x) + g(-x) = cf(x) + g(x) = (cf+g)(x)$.
So $(cf+g)(-x) = (cf+g)(x)$ meaning $cf+g \in W$.
Therefore W is a subspace.

(b) Show that the set $W \cup X$ spans V.

Take an arbitrary $f \in V$. We must show that f is a linear combination of functions in $W \cup X$.

Define $F : \mathbb{R} \to \mathbb{R}$, $F(x) = f(x) + f(-x)$
Define $G : \mathbb{R} \to \mathbb{R}$, $G(x) = f(x) - f(-x)$
Note $F(-x) = F(x)$ so $F \in W$, hence $F \in W \cup X$.
Note $G(-x) = -G(x)$ so $G \in X$, hence $G \in W \cup X$.

Finally observe $f(x) = \frac{1}{2}F(x) + \frac{1}{2}G(x)$ so f is indeed a linear combination of functions in $W \cup X$.

3. Suppose V is a finite-dimensional vector space and $T : V \to V$ is a linear transformation having the property $\text{Range}(T) = \text{Null}(T)$, that is, the range of T and the null space of T are the same subspace.

(a) Show that $\dim(V)$ is an even number.

By Rank Theorem,

$$\begin{aligned}\dim(V) &= \text{rank}(T) + \text{nullity}(T)\\ &= \dim(\text{Range}(T)) + \dim(\text{Null}(T))\\ &= \dim(\text{Null}(T)) + \dim(\text{Null}(T))\\ &= 2\dim(\text{Null}(T))\end{aligned}$$

Thus $\dim(V)$ is even.

(b) Give an example of such a T and V.

If $\text{range}(T) = \text{Null}(T)$ then any element $T\alpha$ in the range is also in the null space, so $TT\alpha = 0$, that is, $T^2\alpha = 0$.

Now let's hunt for such a $T : \mathbb{R}^2 \to \mathbb{R}^2$.

It must be given by matrix multiplication $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \end{bmatrix}$ for a 2×2 matrix A with $A^2 = O$

One such matrix is $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Thus let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 0 \end{bmatrix}$.

Then $\text{Range}(T) = \left\{ \begin{bmatrix} y \\ 0 \end{bmatrix} \mid y \in \mathbb{R} \right\}$

And $\text{Null}(T) = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \mid x \in \mathbb{R} \right\}$

Consequently $\text{Range}(T) = \text{Null}(T)$.