Abstract Algebra II | **Final Exam** | May 9, 2012
MATH 602 | | R. Hammack

Directions: Solve five of the following ten questions.

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1. Give an example of an integral domain R and a nonzero torsion R-module M such that $\text{Ann}(M) = 0$. Prove that if N is a finitely generated torsion R-module, then $\text{Ann}(N) \neq 0$.

2. Show that $p(x) = x^3 + 9x + 6$ is irreducible in $\mathbb{Q}[x]$. Let θ be a root. Find the inverse of $1 + \theta$ in $\mathbb{Q}(\theta)$.

3. Let K/F be an algebraic extension, and let R be a *ring* with $F \subseteq R \subseteq K$.
Show that R is actually a subfield of K.

4. Determine the splitting field and its degree over $\mathbb{Q}$ for $f(x) = x^6 - 4$.

5. Let ζ_n be a primitive n^{th} root of unity, and let d be a divisor of n.
Prove that ζ_n^d is a primitive $(n/d)^{\text{th}}$ root of unity.

6. Prove that $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$ are not isomorphic.

7. Prove that if the Galois group of the splitting field of a cubic over $\mathbb{Q}$ is the cyclic group of order 3, then all the roots of the cubic are real.

9. Construct a finite field with 16 elements. (Be sure to show how both multiplication and addition work.) Find a generator for the multiplicative group. How many different generators are there?

9. Determine the Galois closure of the field $\mathbb{Q}(\sqrt{1 + \sqrt{2}})$ over $\mathbb{Q}$.

10. Determine the Galois group of the polynomial $x^4 - 25$.