# Chapter 9 Polynomial Rings

We now examine rings of polynomials. Historically much of abstract algebra was developed to describe the theory of solutions of polynomial equations. Thus the current topic is central to almost all of abstract algebra, and will set the stage for much of what we will encounter this semester.

At the onset, it is important to recall the following facts.

(Ch 7, Prop 12) An ideal $I \subseteq R$ is maximal $\iff$ $R/I$ is a field

(Ch 7, Prop 13) An ideal $I \subseteq R$ is prime $\iff$ $R/I$ is an integral domain.

(Ch 7, Corr. 14) Every maximal ideal is prime.

(Ch 8, Prop 7) If $R$ is a PID, every prime ideal is maximal.

(Ch 8, Corr. 8) $R[x]$ is a PID $\iff$ $R$ is a field.

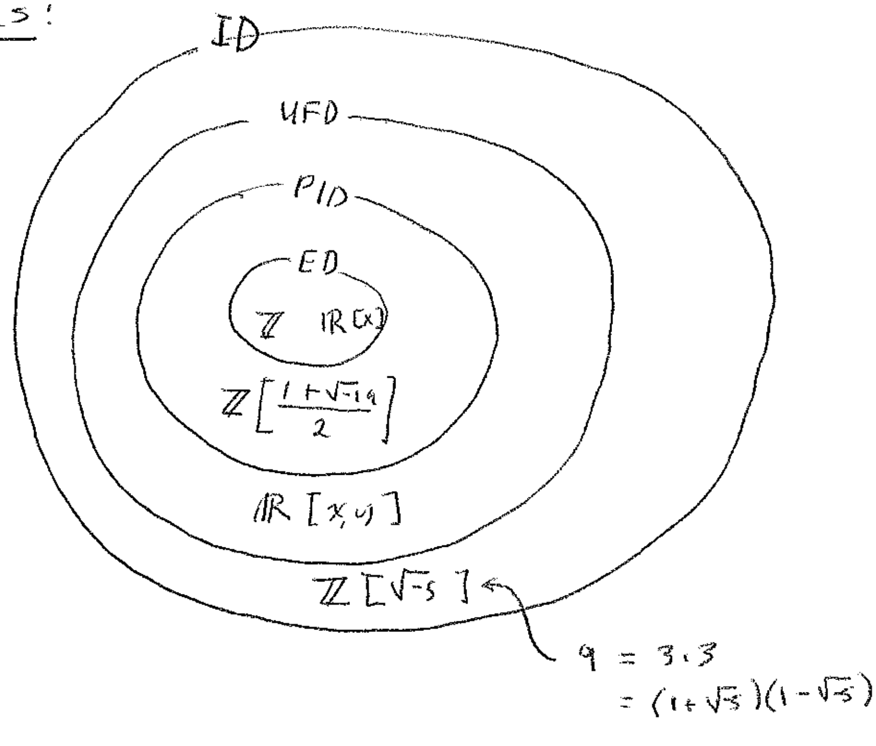
(Ch 8 Prop 1) Every E.D. is a PID

(Ch 8 Theo 14) Every PID is a UFD

(Ch 8 Prop 10) In an I.D. every prime element is ~~maximal~~ irreducible.

(Ch 8 Prop 12) In a UFD every irreducible element is prime

Examples:



# Section 9.1 Definitions and Basic Properties

Convention: $R$ is a commutative ring with $1 \neq 0$.

Definition The ring of polynomials with coefficients in $R$ is
$R[x] = \{ a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \mid a_i \in R, \ n \in \mathbb{Z}^+ \cup \{0\} \}$.
This is a commutative ring under the usual operations of polynomial addition and multiplication.

Suppose $f(x) = a_0 + a_1 x + \cdots + a_n x^n$

- If $a_n \neq 0$ we say $f(x)$ has degree $n$, $\deg f(x) = n$
- If $f(x) = 0$, conventions vary – some say degree is $0$, others $\infty$.
- $a_n$ called leading term; If $a_n = 1$ $f(x)$ is monic.

Note $R = \{ a + 0x \mid a \in R \} \subseteq R[x]$.

Proposition 1 If $R$ is an integral domain, then

1. $\deg p(x)q(x) = \deg p(x) + \deg q(x)$ if $p, q$ are nonzero.
2. Units of $R[x]$ are the units of $R$.
3. $R[x]$ is an integral domain.

However, if $R$ is not an integral domain, all bets are off.

In $\mathbb{Z}/4\mathbb{Z}[x]$:

1. $\underbrace{(2x^2+1)}_{\deg 2}\underbrace{(2x^2+x)}_{\deg 2} = \underbrace{2x^3+2x^2+x}_{\deg 3}$

2. $(2x+1)(2x+1) = 1$, so $2x+1$ is a unit.

3. $(2x+2)(2x+2) = 0$ so $2x+2$ is a zero divisor; $\mathbb{Z}/4\mathbb{Z}[x]$ not an I.D.

Suppose $I \subseteq R$ is an ideal. Then $I \subseteq R \subseteq R[x]$, but $I$ not ideal of $R[x]$.

Consider $(I) \subseteq R[x]$

$$(I) = \left\{ \sum a_i f_i(x) \mid a_i \in I,\ f_i(x) \in R[x] \right\} = I[x] \subseteq R[x]$$

<u>Proposition 2</u> If $I \subseteq R$ is an ideal, then

$$R[x]/(I) \cong R/I\,[x].$$

$$\sum a_i x^i + (I) \longmapsto \sum (a_i + I)\, x^i$$

$$\overline{\sum a_i x^i} \longmapsto \sum \bar{a}_i\, x^i$$

In particular, if $I$ is prime in $R$, then $(I)$ is prime in $R[x]$

<u>Example</u>: $\mathbb{Z}[x]/5\mathbb{Z}[x] \cong \mathbb{Z}/5\mathbb{Z}\,[x]$.

$$\sum a_i x^i + 5\mathbb{Z}[x] \mapsto \sum \bar{a}_i x^i$$

and $5\mathbb{Z}[x]$ is prime ideal in $\mathbb{Z}[x]$.

POLYNOMIALS IN MORE THAN ONE VARIABLE

<u>Example</u> $\mathbb{Z}[x,y] = \left\{ \sum a_{ij} x^i y^j \right\}$

e.g. $5x^3y^2 + 3x + y + 1 + xy \in \mathbb{Z}[x,y]$. etc. $\left\{\begin{array}{l}\text{degree of term } 5x^3y^7 \text{ is } 10 \\ \text{multidegree is } (3,7) \\ \text{degree of entire poly is } 10\end{array}\right.$

Definition $R[x_1, x_2, \ldots x_n] = R[x_1, x_2, \ldots, x_{n-1}][x_n]$

See definitions and examples in text.

(homogeneous polynomials, etc. — we will revisit these definitions as need arises.)

## Section 9.2 Polynomial Rings over Fields I

In high school (and also in Calculus, with partial fractions) you learned how to do long division of polynomials in $\mathbb{R}[x]$. Essentially you were carrying out the division algorithm, and the fact that long division always works means that $\mathbb{R}[x]$ is a E.D. In fact, this is true for any $F[x]$ where $F$ is a field.

**Theorem 3** If $F$ is a field, then $F[x]$ is a E.D. with norm deg. Specifically, if $a(x), b(x) \in F[x]$ and $b(x) \neq 0$ then there are unique $q(x), r(x) \in F[x]$ with

$$a(x) = q(x)\,b(x) + r(x) \quad \text{with } r(x) = 0 \quad \text{or} \quad \deg r(x) < \deg b(x).$$

**Example**

$a(x) = 2x^2 + 3x + 3$
$b(x) = 3x + 4$
in $\mathbb{Z}/5\mathbb{Z}\,[x]$

$$\begin{array}{r|l}
 & 4x + 4 \\
\hline
3x+4 & 2x^2 + 3x + 3 \\
 & 2x^2 + 4x \;\cancel{+3} \\
\hline
 & 2x + 3 \\
 & 2x + 1 \\
\hline
 & 2
\end{array}$$

$$\underbrace{2x^2 + 3x + 3}_{a(x)} = \underbrace{(4x+4)}_{q(x)}\underbrace{(3x+4)}_{b(x)} + \underset{r(x)}{2}$$

[need not be constant in general]

**Corollary 4** If $F$ is a field, $F[x]$ is a PID and a UFD.

(Warning $F[x,y]$ ~~may not be~~ is not a PID - see homework.)

Thus any ideal in $F[x]$ is of form $(f(x)) = \{f(x)g(x) \mid g(x) \in F[x]\}$ for some $f(x) \in F[x]$.

# Understanding the ring structure of quotients of polynomial rings

**Ex** Describe ring structure of $\mathbb{R}[x]/(x^2)$

Note: $x^2 + (x^2) = 0 + (x^2) \rightsquigarrow \overline{x^2} = \overline{0}$

Thus $\mathbb{R}[x]/(x^2)$ is like $\mathbb{R}[x]$, except $x^2 = 0$, $x^3 = 0$, $x^4 = 0 \ldots$

$\mathbb{R}[x]/(x^2) = \{a + bx + (x^2) \mid a, b \in \mathbb{R}\}$

Addition: $a + bx + (x^2) + c + dx + (x^2) = (a+c) + (b+d)x + (x^2)$

Multiplication: $\left(a + bx + (x^2)\right)\left(c + dx + (x^2)\right) = ac + (ad + bc)x + (x^2)$

**Ex** Describe ring structure of $\mathbb{R}[x]/(x^2+1)$

Consider surjective homomorphism $\varphi : \mathbb{R}[x] \longrightarrow \mathbb{C}$

$$f(x) \longmapsto f(i)$$

Check that $\ker \varphi = (x^2+1)$. Thus $\mathbb{R}[x]/(x^2+1) \cong \mathbb{C}$.

Look at it this way:

$\mathbb{R}[x]/(x^2+1)$ is like $\mathbb{R}[x]$ except $x^2+1 = 0$, i.e. $x^2 = -1$.

$\mathbb{R}[x]/(x^2+1) = \{a + bx \mid a, b \in \mathbb{R}\}$

Addition (same as previous ex.)

Mult $(a+bx)(c+dx) = ac + (ad+bc)x - bd = ac - bd + (ad+bc)x$

$(a+bi)(c+di) = \qquad = ac - bd + (ad+bc)i$

**Ex** Describe ring structure of $\mathbb{R}[x,y]/(x-y)$

Like $\mathbb{R}[x,y]$ except $x - y = 0 \rightsquigarrow x = y$

$\mathbb{R}[x,y]/(x-y) \cong \mathbb{R}[x]$.