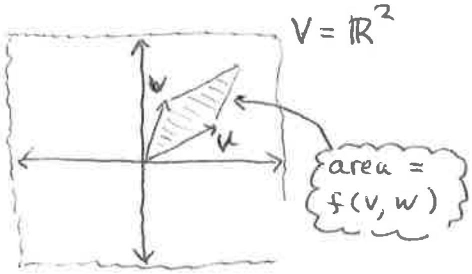
Section 11.4 Determinants

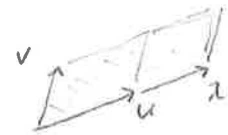
The entire theory of determinants springs from the measurement of volume (and area). To see how, consider the following.



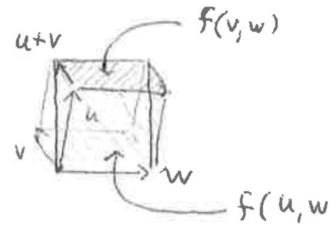
Suppose $f : V \times V \longrightarrow \mathbb{R}$

$f(u,v)$ = area of parallelogram

Properties:

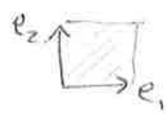


① $f(\lambda u, v) = \lambda f(u,v)$
$\quad f(u, \lambda v) = \lambda f(u,v)$



② $f(u+v, w) = f(u,w) + f(v,w)$
$\quad f(u, v+w) = f(u,v) + f(u,w)$

③ $f(v,v) = 0$



④ $f(e_1, e_2) = 1$

⑤ These imply $f(u,v) = -f(v,u)$

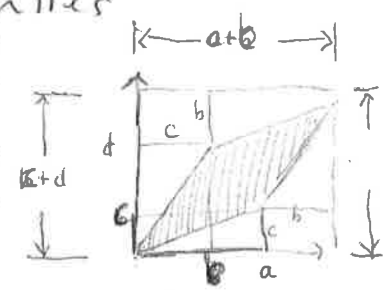
Reason: $0 = f(u+v, u+v) = f(u+v, u) + f(u+v, v)$
$= f(u,u) + f(v,u) + f(u,v) + f(v,v)$
$= f(u,v) + f(v,u)$

$\Rightarrow f(u,v) = -f(v,u)$

Same properties apply to $\mathbb{R}^3$, $\mathbb{R}^n$ by solid geometry.

det measures area

Note det : $\mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ has all these properties



① $\det(\lambda u, v) = \lambda \det(u\ v)$

② $\det(u+v\ w) = \det(u,w) + \det(v,w)$

③ $\det(v\ v) = 0$

④ $\det(e_1\ e_2) = \det I = 1$

⑤ $\det(u\ v) = -\det(v\ u)$

$f\left(\begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix}\right) =$
$= (a+b)(c+d) - ca - bd - cb - cb$
$= ad - bc = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

By above discussion, det on $\mathbb{R}^n$ measures n-D volume

~~Same volume properties apply to~~ $\mathbb{R}^3$ ($\&$ $\mathbb{R}^n$) using solid geometry.

Definitions Suppose V, W are R-modules, and let $\varphi : V \times V \times \cdots \times V \longrightarrow W$.
For each index i, consider map

$$V \longrightarrow W$$
$$x \longmapsto \varphi(\underbrace{v_1\ v_2 \cdots v_{i-1}}_{\text{fixed}}\ \underset{\uparrow \text{ variable}}{x}\ \underbrace{v_{i+1} \cdots v_n}_{\text{fixed}})$$

φ is <u>n-multilinear</u> or <u>multilinear</u> if each such map is an R-module homomorphism. A multilinear map φ is <u>alternating</u>
~~if $\varphi(v_1\ v_2 \cdots v_i\ v_{i+1}\ v_3 \cdots v_n) = -\varphi(v_1\ v_2 \cdots v_{i+1}\ v_i \cdots v_n)$~~
if $\varphi(v_1\ v_2 \cdots v_n) = 0$ whenever $v_i = v_{i+1}$ for some i

Proposition 22 Suppose φ is n-multilinear function on V.

① $\varphi(v_1\ v_2 \cdots v_i\ v_{i+1} \cdots v_n) = -\varphi(v_1\ v_2 \cdots v_{i+1}\ v_i \cdots v_n)$

② If $\pi \in S_n$ then $\varphi(v_{\pi(1)}\ v_{\pi(2)} \cdots v_{\pi(n)}) = \varepsilon(\pi)\ \varphi(v_1\ v_2 \cdots v_n)$

③ If $v_i = v_j$ for $i \neq j$ Then $\varphi(v_1\ v_2 \cdots v_n) = 0$.

④ $\varphi(v_1, v_2, \cdots, v_i, \cdots, v_n) = \varphi(v_1, v_2, \cdots, v_i + \alpha v_j, \cdots, v_n)$

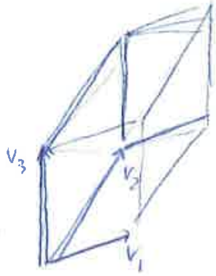
Definition An <u>$n \times n$ determinant function</u> on R is any function

$$\det : M_{n\times n}(R) \longrightarrow R$$
i.e. $$\det : R \times R \times \cdots \times R \longrightarrow R$$

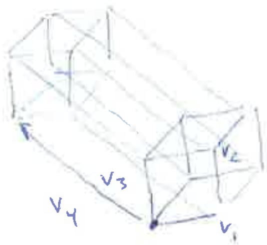
for which det is n-multilinear, alternating and $\det(I) = 1$.

Theorem There is exactly one $n \times n$ determinant function on R. ~~If $A = [$~~ It has the following form

$$\det\left(\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}\right) = \sum_{\pi \in S_n} \varepsilon(\pi)\ \alpha_{\pi(1)1}\ \alpha_{\pi(2)2} \cdots \alpha_{\pi(n)n}$$



$\det(v_1\ v_2\ v_3) = \pm$ volume of parallelopiped



$\det(v_1\ v_2\ v_3\ v_4) = \pm$ hyper volume of parallelepiped.

Determinant Functions on V, (a vector space over F)

Suppose V has basis $\mathcal{B} = \{b_1\ b_2 \cdots b_n\}$
So V^* has basis $\mathcal{B}^* = \{b_1^*\ b_2^* \cdots b_n^*\}$

Then $\varphi(x, y) = b_i^*(x)\, b_j^*(y)$ is a bilinear function $\varphi : V \times V \to F$

Any bilinear function on V has form $\varphi(x, y) = \sum a_{ij}\, b_i(x)\, b_j(y)$

{Meaningful to multiply vectors in V^* ! }

Fact: $V^* \otimes V^* \cong$ bilinear forms on V

$$\sum a_{ij}\, b_i^* \otimes b_j^* \longmapsto \sum a_{ij}\, b_i^*(*)\, b_j^*(\_)$$

Fact: $V^* \otimes V^* \otimes \cdots \otimes V^* \cong$ n-multilinear forms on V.

Fact Subspace $\{$alternating multilinear forms on $V\} \subseteq \underbrace{V^* \otimes V^* \cdots \otimes V^*}_{n}$

is one-dimensional. Spanned by

$$\det = \frac{1}{n!} \sum_{\pi \in S_n} \varepsilon(\pi)\, b^*_{\pi(1)} \otimes b^*_{\pi(2)} \otimes b^*_{\pi(3)} \otimes \cdots \otimes b^*_{\pi(n)}$$

See Section 11.5 in *Tensor Algebras*.