Section 11.2 The Matrix of a Linear Transformation (Quick review)

Set up: Finite dimensional vector spaces over F:

V, basis $\mathcal{B} = \{v_1\ v_2 \cdots v_n\}$

$v = \sum_{j=1}^{n} \beta_i v_i \in V$ has column representation $[v]_{\mathcal{B}} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}$

W, basis $\mathcal{E} = \{w_1\ w_2 \cdots w_m\}$

$w = \sum_{i=1}^{m} \gamma_i w_i \in W$ has column reprensentation $[w]_{\mathcal{E}} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_m \end{bmatrix}$

Consider linear transformation $\varphi : V \longrightarrow W$. Then

$$\varphi(v_j) = \sum_{i=1}^{n} \alpha_{ij} w_i$$

Matrix of φ relative to $\mathcal{B}$ and $\mathcal{E}$ is $M_{\mathcal{B}}^{\mathcal{E}}[\varphi] = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{bmatrix}$

Has property $M_{\mathcal{B}}^{\mathcal{E}}[\varphi]\,[v]_{\mathcal{B}} = [\varphi(v)]_{\mathcal{E}}$

$\{ Mv = \varphi(v) \}$ ← matrix represents φ as matrix multiplication.

Theorem 10 Map $\varphi \longmapsto M_{\mathcal{B}}^{\mathcal{E}}[\varphi]$ is isomorphism $\mathrm{Hom}_F(V,W) \longrightarrow M_{n\times m}(F)$.

Corollary 11 $\dim \mathrm{Hom}_F(V,W) = \dim(V) \cdot \dim(W)$.

Theorem 12 Given

$$\begin{array}{ccccccl} U & \xrightarrow{\psi} & V & \xrightarrow{\varphi} & W & & \leftarrow (\text{spaces}) \\ (\mathcal{D}) & & (\mathcal{B}) & & (\mathcal{E}) & & \leftarrow (\text{bases}) \end{array}$$

we have $M_{\mathcal{D}}^{\mathcal{E}}[\varphi \circ \psi] = M_{\mathcal{B}}^{\mathcal{E}}[\varphi]\, M_{\mathcal{D}}^{\mathcal{B}}[\psi]$ ← (matrix mult.)

Theorem 14 Given space V with basis $\mathcal{B}$, map $\varphi \mapsto M_{\mathcal{B}}^{\mathcal{B}}[\varphi]$ is a ring isomorphism $\mathrm{Hom}_F(V,V) \longrightarrow M_{n\times n}[F]$.

Also $GL(V) \cong GL_n(F)$ and $M_{\mathcal{B}}^{\mathcal{B}}[\varphi^{-1}] = \left(M_{\mathcal{B}}^{\mathcal{B}}[\varphi]\right)^{-1}$

Recall Column rank of a matrix is max # of lin. ind. columns
Row rank " " " " " " " " " rows

Definition The rank of a transformation $\varphi : V \longrightarrow W$ is
$\mathrm{rank}(\varphi) = \dim(\varphi(V)) = \text{col. rank}\left(M_{\mathcal{B}}^{\mathcal{E}}(\varphi)\right)$

Similar Matrices

$n \times n$ matrices A & B are <u>similar</u> if there is an invertible matrix P for which $P^{-1}AP = B$.

What does this mean? Here's how to understand it.

Suppose V has two bases $\mathcal{B} = \{v_1\ v_2 \cdots v_n\}$ and $\mathcal{B}' = \{v_1'\ v_2' \cdots v_n'\}$. Consider L.T. $\varphi : V \to V$.

$$
\begin{tikzcd}
V \arrow[r, "\varphi"] \arrow[d, "id"] & V \; \} \mathcal{B} \\
V \arrow[r, "\varphi"] & V \; \} \mathcal{B}' \arrow[u, "id^{-1}"']
\end{tikzcd}
$$

Let $\mu : V \to V$ be the L.T. taking $\mathcal{B}$ to $\mathcal{B}'$. Then:

$$\varphi = id^{-1}\, \varphi\ id$$

$$M_{\mathcal{B}}^{\mathcal{B}}[\varphi] = M_{\mathcal{B}}^{\mathcal{B}}[id^{-1} \varphi\ id]$$

$$M_{\mathcal{B}}^{\mathcal{B}}[\varphi] = M_{\mathcal{B}}^{\mathcal{B}'}[id]^{-1}\, M_{\mathcal{B}'}^{\mathcal{B}'}[\varphi]\, M_{\mathcal{B}}^{\mathcal{B}'}[id]$$

Thus $M_{\mathcal{B}}^{\mathcal{B}}[\varphi]$ is similar to $M_{\mathcal{B}'}^{\mathcal{B}'}[\varphi]$

Note: $M_{\mathcal{B}}^{\mathcal{B}'}[id] \neq \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$ in general

<u>Conclusion</u> Matrices for φ relative to different bases are similar.

Section 11.3 Dual Spaces

Definition If V is a vector space over F, then its dual space is
$$V^* = \mathrm{Hom}_F(V, F).$$
Any $f \in V^*$ is called a linear functional.

Example $V = \mathbb{R}^2$. Given a 1×2 matrix $[a, b]$ let $f : \mathbb{R}^2 \to \mathbb{R}$ be the map $\varphi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = [a, b]\begin{bmatrix} x \\ y \end{bmatrix} = ax + by$. (mult by $[a,b]$). Then $f \in V^*$ and any liner map $f : \mathbb{R}^2 \to \mathbb{R}$, $f \in V^*$, can be described by such a matrix. Thus $V^* \cong \{[a, b] \mid a, b \in \mathbb{R}\} \cong \mathbb{R}^2 = V$.

Proposition 18 Let $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ be a basis for a finite-dimensional vector space over F. For each $v_i \in \mathcal{B}$, define $v_i^* \in V^*$:
$$v_i^*(v_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$
so $v_i^*\left(\sum_{j=1}^{n} \alpha_j v_j\right) = \alpha_i$. Then $v_i^* \in V$ for each i, and $\mathcal{B}^* = \{v_1^*, v_2^*, \cdots v_n^*\}$ is a basis for V^*. In particular, $V \cong V^*$ and the map $\varphi : V \to V^*$, $\varphi(\sum \alpha_j v_j) = \sum \alpha_j v_j^*$ is an isomorphism.

Definition $\mathcal{B}^*$ is called the dual basis to $\mathcal{B}$.

Note There is always an injective linear map $\varphi : V \longrightarrow V^*$. Just take a basis $\mathcal{B}$ of V and its dual basis $\mathcal{B}^*$ and define φ as in Proposition 18.

Observations

1. Map φ is not "natural," that is, it depends on choice of $\mathcal{B}$. Different $\mathcal{B}$ will yield different φ

2. Although $V \cong V^*$ when V is finite dimensional (φ is also surjective) $\dim(V) < \dim(V^*)$ when V is infinite-dimensional. Thus always $V \not\cong V^*$ for infinite-dimensional spaces. (See Exercise 4)

3. It makes sense to form the dual of V^*, i.e. V^{**}, the double dual. There is a "natural" (i.e. not dependent on a basis) injective linear map $\varphi : V \longrightarrow V^{**}$

Theorem 19 There is a natural injective linear transformation

$$\varphi : V \longrightarrow V^{**}$$

Proof Define φ as follows: If $v \in V$, then $\varphi(v) \in (V^*)^*$ so $\varphi(v)$ is a linear map $\varphi(v) : V^* \to F$. Define $\varphi(v)(f) = f(v)$.

Note $\varphi(v) \in (V^*)^*$ because $\varphi(v)(f+\alpha g) = (f+\alpha g)(v) = f(v) + \alpha g(v) = \varphi(v)(f) + \alpha \varphi(v)(g)$.

Note $\varphi : V \to V^{**}$ is linear because: $\varphi(v+\alpha w)(f) = f(v+\alpha w)$

$= f(v) + \alpha f(w) = \varphi(v)(f) + \alpha \varphi(w)(f) = (\varphi(v) + \alpha \varphi(w))(f)$.

Therefore $\varphi(v+\alpha w) = \varphi(v) + \alpha \varphi(w)$

It remains to show that φ is injective — ie if $v \neq 0$, then $\varphi(v) \neq 0$. Suppose $v \neq 0$. Take basis $\mathcal{B} = \{v, \dots\}$ of V. Then $v^* \in V^*$. Now $\varphi(v)(v^*) = 1$, so $\varphi(v)$ cant be $0 \in V^{**}$. Thus $\ker \varphi = 0$ and φ is injective ■

Given L.T. $V \xrightarrow{\varphi} W$, $v \longmapsto \varphi(v)$

Get L.T. $V^* \xleftarrow{\varphi^*} W^*$, $f \circ \varphi \longleftarrow f$

Linear: Claim $\varphi^*(f+\alpha g) = \varphi^*(f) + \alpha \varphi^*(g)$

$\varphi^*(f+\alpha g)(v) = (f+\alpha g) \circ \varphi(v) =$

$(f+\alpha g)(\varphi(v)) = f(\varphi(v)) + \alpha g(\varphi(v))$

$= f \circ \varphi(v) + \alpha g \circ \varphi(v) = (\varphi^*(f) + \alpha \varphi^*(g))\, v$

Theorem 20 Given L.T. $\varphi : V \to W$, $\exists$ L.T. $\varphi^* : W^* \to V^*$ defined as $\varphi^*(f) = f \circ \varphi$. If $\mathcal{B}$ and $\mathcal{E}$ are bases for V and W, resp, then

$$M^{\mathcal{B}^*}_{\mathcal{E}^*}[\varphi^*] = \left(M^{\mathcal{E}}_{\mathcal{B}}[\varphi]\right)^T \quad \text{(transpose)}.$$

Corollary 21 For any matrix A, the row-rank and column-rank are the same.