

## Section 10.4 Tensor Products of Modules (Continued)

### Recall Extension by Scalars.

If  $N$  is a (left)  $R$ -module and  $S \subseteq R$  is a subring of  $R$ , then  $S \otimes_R N$  is the "closest match to  $N$  as an  $S$ -module."

Construction:  $S \otimes_R N = F(S \times N) / \langle H \rangle$ , where

$$H = \left\{ (s+s', n) - (s, n) - (s', n) \mid s, s' \in S, n \in N \right\} \cup \\ \left\{ (s, n+n') - (s, n) - (s, n') \mid s \in S, n, n' \in N \right\} \cup \\ \left\{ (sr, n) - (s, rn) \mid s \in S, r \in R, n \in N \right\}$$

Notation:  $(s, n) + H = s \otimes n \leftarrow$  "simple tensor"

Thus:  $S \otimes_R N = \left\{ \sum_{\text{finite}} s_i \otimes n_i \right\}$

Properties:  $(s+s') \otimes n = s \otimes n + s' \otimes n$

$$s \otimes (n+n') = s \otimes n + s \otimes n'$$

$$sr \otimes n = s \otimes rn$$

Check that  $S \otimes_R N$  is an  $S$ -module under action

$$s \cdot \left( \sum s_i \otimes m_i \right) = \sum ss_i \otimes m_i$$

Example  $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}_6 = \{0\}$  because

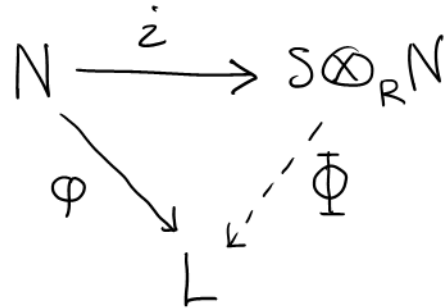
$$\frac{a}{b} \otimes n = \frac{a6}{b6} \otimes n = \frac{a}{b6} \otimes 6n = \frac{a}{6b} \otimes 0 = \frac{a}{6b} \otimes 0 \cdot 0 =$$

$$\frac{a \cdot 0}{6b} \otimes 0 = 0 \otimes 0. \quad (\text{All simple tensors are zero.})$$

Theorem 8 Let  $\iota: N \rightarrow S \otimes_R N$  be the  $R$ -module homomorphism  $\iota(n) = 1 \otimes n$ . Suppose  $L$  is any  $S$ -module and  $\varphi: N \rightarrow L$  is an  $R$ -module homomorphism. Then there exists a unique  $S$ -module homomorphism  $\bar{\varphi}: S \otimes_R N \rightarrow L$  for which

$$\bar{\varphi}(\iota(n)) = \varphi(n).$$

$$\text{i.e. } \bar{\varphi}(1 \otimes n) = \varphi(n).$$



Conversely, if  $\bar{\varphi}: S \otimes_R N \rightarrow L$  is an  $S$ -module homomorphism, then  $\bar{\varphi} \circ \iota$  is an  $R$ -module homomorphism.

Example  $R \otimes_R N = N$

Proof Apply theorem to  $\varphi: N \xrightarrow{\text{id}} N$ .

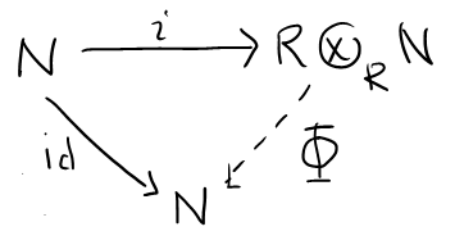
(identity) Get  $R$ -module homomorphism

$$\bar{\varphi}: R \otimes_R N \rightarrow N. \text{ Note } \bar{\varphi} \text{ is surjective;}$$

If  $n \in N$  then  $\bar{\varphi}(\iota(n)) = \varphi(n) = \text{id}(n) = n$ .

Injective: Note  $\bar{\varphi}(\sum r_i \otimes n_i) = \bar{\varphi}(\sum 1 \otimes r_i n_i) = \bar{\varphi}(1 \otimes \sum r_i n_i) = \sum r_i n_i$ . If this is zero, then so is  $\bar{\varphi}(1 \otimes \sum r_i n_i) = \bar{\varphi}(\sum r_i \otimes n_i)$

Therefore kernel is zero

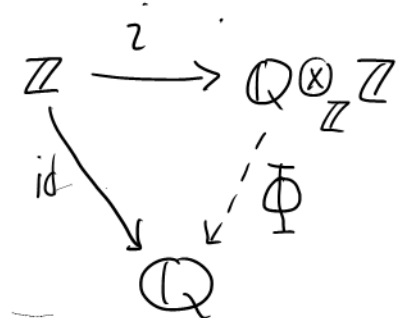


Example  $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Q}$

Apply theorem to get  $\bar{\varphi}: \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z} \rightarrow \mathbb{Q}$ .

Surjective: Take  $\frac{a}{b} \in \mathbb{Q}$ . Diagram says  $\bar{\varphi}(1 \otimes a) = a$ . Then  $\bar{\varphi}(\frac{1}{b} \otimes a) = \frac{1}{b} \bar{\varphi}(1 \otimes a) = \frac{1}{b} a = \frac{a}{b}$ .

Check injectivity as before.



## General Construction (Informal Picture)

Suppose  $R$  is commutative,  $M, N$  left  $R$ -modules.

Define right  $R$ -action on  $M$  as  $mr = rm$ .

Define  $M \otimes_R N$  as follows:

① Form free abelian group  $F(M \times N)$

②  $H = \{ (m+m', n) - (m, n) - (m', n), \\ (m, n+n') - (m, n) - (m, n'), \\ (mr, n) - (m, rn) \mid m, m' \in M, n, n' \in N, r \in R \}$

③ Put  $M \otimes_R N = F(M \times N) / \langle H \rangle$

Let  $(m, n) + \langle H \rangle = m \otimes n$

so  $M \otimes_R N = \{ \sum m_i \otimes n_i \}$

This is  $R$ -module with action  $r \sum m_i \otimes n_i = \sum rm_i \otimes n_i$

Properties  $(m+m') \otimes n = m \otimes n + m' \otimes n$   
 $m \otimes (n+n') = m \otimes n + m \otimes n'$   
 $mr \otimes n = m \otimes rn$

Definition  $\varphi: M \times N \rightarrow L$  is  $R$ -bilinear if  $\begin{cases} \varphi(m+m', n) = \varphi(m, n) + \varphi(m', n) \\ \varphi(m, n+n') = \varphi(m, n) + \varphi(m, n') \\ \varphi(rm, n) = r \varphi(m, n) \\ \varphi(m, rn) = r \varphi(m, n) \end{cases}$

### Corollary 12

The map  $z: M \times N \rightarrow M \otimes_R N$  where  $z(m, n) = m \otimes n$  is  $R$ -bilinear. Also, if  $\varphi: M \times N \rightarrow L$  is any  $R$ -bilinear map then there is a unique  $R$ -module homomorphism  $\Phi: M \otimes_R N \rightarrow L$ :

$$M \times N \xrightarrow{z} M \otimes_R N$$

$$\begin{array}{ccc} & & \Phi \\ & \searrow \varphi & \swarrow \\ & L & \end{array}$$

with  $\Phi \circ z = \varphi$ , that is,  $\Phi(m \otimes n) = \varphi(m, n)$

Bijection:  $\left\{ \begin{array}{l} R\text{-bilinear maps} \\ \varphi: M \times N \rightarrow L \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} R\text{-module homomorphisms} \\ \Phi: M \otimes_R N \rightarrow L \end{array} \right\}$