Section 10.4 Tensor Products of Modules (Continued)

Recall Extension by scalars.

If N is a (left) R-module and $S \subseteq R$ is a subring of R, then $S \otimes_R N$ is the "closest match to N as an S-module."

Construction: $S \otimes_R N = F(S \times N)/\langle H \rangle$, where

$$H = \{ (s+s', n) - (s,n) - (s',n) \mid s, s' \in S,\ n \in N \} \cup$$
$$\{ (s, n+n') - (s,n) - (s,n') \mid s \in S,\ n, n' \in N \} \cup$$
$$\{ (sr, n) - (s, rn) \mid s \in S,\ r \in R,\ n \in N \}$$

Notation: $(s,n) + H = s \otimes n \longleftarrow$ "simple tensor"

Thus: $S \otimes_R N = \left\{ \sum_{\text{finite}} s_i \otimes n_i \right\}$

Properties:

$$(s+s') \otimes n = s \otimes n + s' \otimes n$$
$$s \otimes (n+n') = s \otimes n + s \otimes n'$$
$$sr \otimes n = s \otimes rn$$

Check that $S \otimes_R N$ is an S-module under action

$$s . \left(\sum s_i \otimes m_i \right) = \sum s s_i \otimes m_i$$

Example $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}_6 = \{0\}$ because

$$\frac{a}{b} \otimes n = \frac{a6}{b6} \otimes n = \frac{a}{b6} \otimes 6n = \frac{a}{6b} \otimes 0 = \frac{a}{6b} \otimes 0.0 =$$
$$\frac{a \cdot 0}{6b} \otimes 0 = 0 \otimes 0.$$ (All simple tensors are zero)

__Theorem 8__ Let $i : N \longrightarrow S \otimes_R N$ be the R-module homomorphism $i(n) = 1 \otimes n$. Suppose L is any S-module and $\varphi : N \longrightarrow L$ is an R-module homomorphism. Then there exists a unique S-module homomorphism $\Phi : S \otimes_R N \longrightarrow L$ for which

$\Phi(i(n)) = \varphi(n)$.

i.e. $\Phi(1 \otimes n) = n$.

$$\begin{tikzcd} N \arrow[r, "i"] \arrow[dr, "\varphi"'] & S \otimes_R N \arrow[d, dashed, "\Phi"] \\ & L \end{tikzcd}$$

Conversely, if $\Phi : S \otimes_R N \longrightarrow L$ is an S-module homomorphism, then $\Phi \circ i$ is an R-module homomorphism.

__Example__ $R \otimes_R N = N$

__Proof__ Apply theorem to $\varphi : N \xrightarrow{id} N$. (identity) Get R-module homomorphism $\Phi : R \otimes_R N \longrightarrow N$. Note Φ is surjective:

$$\begin{tikzcd} N \arrow[r, "i"] \arrow[dr, "id"'] & R \otimes_R N \arrow[d, dashed, "\Phi"] \\ & N \end{tikzcd}$$

If $n \in N$ then $\Phi(i(n)) = \varphi(n) = id(n) = n$.

Injective: Note $\Phi(\sum r_i \otimes n_i) = \Phi(\sum 1 \otimes r_i n_i) = \Phi(1 \otimes \sum r_i n_i)$
$= \sum r_i n_i$. If this is zero, then so is $\Phi(1 \otimes \sum r_i n_i) = \Phi(\sum r_i \otimes n_i)$
Therefore kernel is zero

__Example__ $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Q}$

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "i"] \arrow[dr, "id"'] & \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z} \arrow[d, dashed, "\Phi"] \\ & \mathbb{Q} \end{tikzcd}$$

Apply theorem to get $\Phi : \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z} \to \mathbb{Q}$.

Surjective: Take $\frac{a}{b} \in \mathbb{Q}$. Diagram says $\Phi(1 \otimes a) = a$. Then $\Phi(\frac{1}{b} \otimes a) =$
$\frac{1}{b} \Phi(1 \otimes a) = \frac{1}{b} a = \frac{a}{b}$.

Check injectivity as before.

General Construction (Informal Picture)

Suppose R is commutative, M, N left R-modules.
Define right R-action on M as $mr = rm$.

Define $M \otimes_R N$ as follows:

① Form free abelian group $F(M \times N)$

② $$H = \{ (m+m', n) - (m,n) - (m', n), \; (m, n+n') - (m\,n) - (m, n'), \; (mr, n) - (m, rn) \mid m, m' \in M,\ n, n' \in N,\ r \in R \}$$

③ Put $M \otimes_R N = F(M \times N)/\langle H \rangle$

Let $(m,n) + \langle H \rangle = m \otimes n$

so $M \otimes_R N = \{ \sum m_i \otimes n_i \}$

This is R-module with action $r \sum m_i \otimes n_i = \sum r m_i \otimes n_i$

Properties

$$(m+m') \otimes n = m \otimes n + m' \otimes n$$
$$m \otimes (n+n') = m \otimes n + m \otimes n'$$
$$mr \otimes n = m \otimes rn$$

Definition $\varphi : M \times N \to L$ is R-bilinear if
$$\begin{cases} \varphi(m+m', n) = \varphi(m,n) + \varphi(m', n) \\ \varphi(m, n+n') = \varphi(m,n) + \varphi(m, n') \\ \varphi(rm, n) = r\,\varphi(m,n) \\ \varphi(m, rn) = r\,\varphi(m,n) \end{cases}$$

Corollary 12

The map $i : M \times N \longrightarrow M \otimes N$ where $i(m,n) = m \otimes n$ is R-bilinear. Also, if $\varphi : M \times N \to L$ is any R-bilinear map then there is a unique R-module homomorphism Φ:

$$\begin{tikzcd} M \times N \arrow[rr, "i"] \arrow[dr, "\varphi"'] & & M \otimes_R N \arrow[dl, dashed, "\Phi"] \\ & L & \end{tikzcd}$$

with $\Phi \circ i = \varphi$, that is, $\Phi(m \otimes n) = \varphi(m,n)$

Bijection: $\left\{ \begin{array}{l} R\text{-bilinear maps} \\ \varphi : M \times N \to L \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} R\text{-module homomorphisms} \\ \Phi : M \otimes_R N \longrightarrow L \end{array} \right\}$