Free Abelian Groups

Definition Given a set A, the $\mathbb{Z}$-module $F(A)$ is called the free abelian group on A. $F(A) = \langle a_1, a_2, \dots a_n \mid a_i + a_j = a_j + a_i \rangle$

Intuitive Idea If $A = \{a_1 a_2 \dots a_n\}$, then $F(A) = \{r_1 a_1 + r_2 a_2 + \dots + r_n a_n \mid r_i \in \mathbb{Z}\}$ where the a_i are "unlike terms" that cannot be combined.

$$\text{Addition: } \sum r_i a_i + \sum r_i' a_i = \sum (r_i + r_i') a_i$$

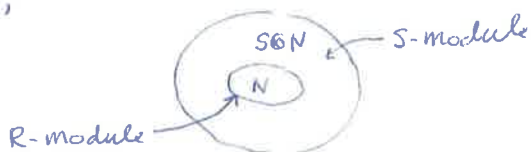
Therefore $F(A) \cong \bigoplus_{i=1}^{n} \mathbb{Z} = \mathbb{Z}^n$.

This works even if A is infinite. In that case

$$F(A) = \Big\{ \sum_{\text{finite}} r_i a_i \;\Big|\; r_i \in \mathbb{Z},\ a_i \in A \Big\} \cong \bigoplus_{a \in A} \mathbb{Z}$$

Section 10.4 Tensor products of modules

Goals ① If N is an R-module and R is a subring of S, construct $S \otimes N$, an S-module that "extends" N to an S-module



② Given R-modules M, N, construct $M \otimes N$ an R-module that allows for "multiplication" of elements of M by those of n.

① Extension of Scalars

Suppose M is an R-module and $R \subseteq S$.

Goal: Construct $S \otimes N$, the "closest match" to N as a S-module.

Plan: $\quad sn \xleftrightarrow{\text{encode}} (s, n) \in S \times N$

Step 1 Form free abelian group $F(S \times N)$

$$\underbrace{\sum s_i n_i}_{\text{want}} \longleftrightarrow \underbrace{\sum (s_i, n_i) \in F(S \times N)}_{\text{have}}$$

Step 2

Want module properties:

$(s+s')n = sn + s'n$ $\quad (s+s')n - sn - s'n = 0$

$s(n+n') = sn + sn'$ $\quad s(n+n') - sn - sn' = 0$ $\longleftrightarrow$

$(sr)n = s(rn)$ $\quad (sr)n - s(rn) = 0$

Need to force this:

$(s+s', n) - (s,n) - (s', n) = 0$

$(s, n+n') - (s,n) - (s,n') = 0$

$(sr, n) - (s, rn) = 0$

Let $H = \{ (s+s', n) - (s,n) - (s',n) \mid s, s' \in S,\ n \in N \} \cup$
$\{ (s, n+n') - (s,n) - (s, n') \mid s \in S,\ n, n' \in N \} \cup$
$\{ (sr, n) - (s, rn) \mid s \in S,\ r \in R,\ n \in N \}$.

Step 3 The tensor product of S and N over R.

Put $S \otimes_R N = F(S \times N) / \langle H \rangle$

Notation: $(s, m) + H = s \otimes n$ ⟵ "simple tensor"

Thus $S \otimes_R N = \left\{ \sum_{\text{finite}} s_i \otimes m_i \right\}$ (elements called "tensors")

Ideally we have $N \subseteq S \otimes_R N$ by $n \longleftrightarrow 1 \otimes n$

but $S \otimes_R N$ has extra stuff $s \otimes n$ corresponding to action of s on n.

Properties: $(s+s') \otimes n = s \otimes n + s' \otimes m$
$s \otimes (m + n') = s \otimes n + s \otimes m'$
$(sr \otimes n) = (s \otimes rn)$

S-module properties

Check $S \otimes_R M$ is an S-module with action

$$s\left(\sum s_i \otimes m_i\right) = \sum s s_i \otimes m_i$$

Ex $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}_6 = \{1 \otimes 0\}$ $\quad \frac{a}{b} \otimes m = \frac{6a}{6b} \otimes m = \frac{1}{6b} \otimes 6am = \frac{1}{6b} \otimes 0 = \frac{1}{6b} \otimes$

Theorem 8 Let $i : N \to S \otimes_R N$ be R-module homo. $i(n) = 1 \otimes n$. Suppose L is any S-module and $\varphi : N \to L$ is R-module homo.

$$\begin{tikzcd} N \arrow[r, "i"] \arrow[dr, "\varphi"'] & S \otimes_R N \arrow[d, "\Phi"] \\ & L \end{tikzcd}$$

Then $\exists$ unique S-module homo $\Phi : S \otimes_R N \longrightarrow L$ for which $\Phi(i(n)) = \varphi(n)$. Conversely, if $\Phi : S \otimes_R N \to L$ is an S-module homomorphism, Then $\varphi = \Phi \circ i$ is an R-module homomorphism.

Example $R \otimes_R N = N$ for any R-module.

Proof

$$\begin{tikzcd} N \arrow[r, "i"] \arrow[dr, "id"'] & R \otimes_R N \arrow[d, "\Phi"] \\ & N \end{tikzcd}$$

This means Φ is surjective and i injective.

Also i surjective. Given $\sum r_i \otimes n_i \in R \otimes_R N$, this is $\sum 1 \otimes r_i n_i = 1 \otimes \sum r_i n_i = i(\sum r_i n_i)$

Therefore $\begin{tikzcd} N \arrow[r, bend left, "i"] & R \otimes_R N \arrow[l, bend left, "\Phi"] \end{tikzcd}$ $\quad \Phi \circ i = id$

Example $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Q}$

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "i"] \arrow[dr, "\varphi d"'] & \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z} \arrow[d, "\Phi"] \\ & \mathbb{Q} \end{tikzcd}$$

Φ surjective

Take $\frac{a}{b} \in \mathbb{Q}$. Diagram: $\Phi(1 \otimes a) = a$

Th. $\frac{1}{b} \Phi(1 \otimes a) = \frac{1}{b} a$

$\Phi(\frac{1}{b} \otimes a) = \frac{a}{b}$

Φ injective: