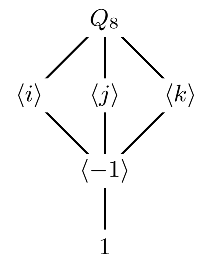
Abstract Algebra I | MATH 601 Midterm | November 1, 2011

Name:____________________ R. Hammack Score:________

1. Short Answer (8 points each)

   (a) Draw the subgroup lattice for Q_8.



   (b) Find the order of $\overline{30}$ in $\mathbb{Z}/54\mathbb{Z}$.
   Since $\overline{30} = 30\,\overline{1}$, Proposition 5 (Chapter 3) gives $|\overline{30}| = |30\,\overline{1}| = \frac{54}{\gcd(54,30)} = \frac{54}{6} = 9$.
   Computing directly, $\langle \overline{30} \rangle = \{\overline{0}, \overline{30}, \overline{6}, \overline{36}, \overline{12}, \overline{42}, \overline{18}, \overline{48}, \overline{24}\}$.

   (c) State the class equation.
   Let $g_1, g_2, \ldots, g_k$ be representatives from the conjugacy classes of G that have more than one element. Then

$$|G| = |Z(G)| + \sum_{i=1}^{k} |G : C_G(g_i)|.$$

   (d) Write down the elements of a Sylow 2-subgroup of A_4.
   $V = \{1, (12)(34), (13)(24), (14)(23)\}$

   (e) Give an example of a non-abelian group that is simple.
   The smallest example is A_5.

2. Suppose $n \geq 3$. Show that the set $A = \{x \in D_{2n} \,|\, x^2 = 1\}$ is not a subgroup of D_{2n}.

   Consider the usual notation $D_{2n} = \{1, r, r^2, \ldots, r^{n-1}, s, sr, sr^2, \ldots, sr^{n-1}\}$.
   Certainly we have $1^2 = 1$ and $s^2 = 1$, but also observe that

$$(sr^k)^2 = (sr^k)(sr^k) = (sr^k)(r^{-k}s) = sr^k r^{-k} s = ss = 1.$$

   This gives us at least $n+1$ elements $1, s, sr, sr^2, \ldots, sr^{n-1}$ whose square is 1.
   Now, not every element of D_{2n} has 1 as a square, since $r^2 \neq 1$.
   Therefore $n+1 \leq |A| < 2n$. If A were a subgroup, its order would have to divide $|D_{2n}| = 2n$, but that's impossible because $n < |A| < 2n$. Conclusion: A is not a subgroup.

3. Prove the multiplicative group $\mathbb{Q}^+$ of positive rational numbers is generated by the set $A = \left\{ \frac{1}{p} \,|\, p \text{ is prime} \right\}$.

   Proof: First we are going to show that any reciprocal $\frac{1}{m}$ of a positive integer m is a product of powers of elements of A. Let m have prime factorization $m = p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k}$. Then

$$\frac{1}{m} = \left(\frac{1}{p_1}\right)^{x_1} \left(\frac{1}{p_2}\right)^{x_2} \cdots \left(\frac{1}{p_k}\right)^{x_k}$$

   is a product of powers of elements of A, so it belongs to $\langle A \rangle$. Similarly, if $n = p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k}$, then

$$n = \left(\frac{1}{p_1}\right)^{-x_1} \left(\frac{1}{p_2}\right)^{-x_2} \cdots \left(\frac{1}{p_k}\right)^{-x_k}$$

   so it follows that any positive integer belongs to $\langle A \rangle$.

   Finally, consider an arbitrary $\frac{n}{m} \in \mathbb{Q}^+$. Because $n, \frac{1}{m} \in A$ (as established above), we have $\frac{n}{m} = n\frac{1}{m} \in \langle A \rangle$. This shows $\mathbb{Q}^+ \leq \langle A \rangle$. On the other hand, it is obvious that $\langle A \rangle \leq \mathbb{Q}^+$. Therefore $\mathbb{Q}^+ = \langle A \rangle$. ■

4. Prove that if $G/Z(G)$ is cyclic, then G is abelian.

Proof: Suppose $G/Z(G)$ is cyclic.
Then for some $a \in G$ we have $G/Z(G) \cong \langle aZ(G) \rangle = \{Z(G), aZ(G), a^2Z(G), a^3Z(G), \ldots, a^{n-1}Z(G)\}$, with $a^nZ(G) = Z(G)$. Because the cosets in $G/Z(G)$ form a partition of G, any two elements $x, y \in G$ can be written as $x = a^kz_1$ and $y = a^\ell z_2$ for appropriate powers k, ℓ and $z_1, z_2 \in Z(G)$. Then

$$xy = (a^kz_1)(a^\ell z_2) = a^kz_1a^\ell z_2 = a^ka^\ell z_1z_2 = a^\ell a^kz_2z_1 = (a^\ell z_2)(a^kz_1) = yx.$$

Therefore G is abelian. ■

5. Prove that if $|G : H| = 2$, then $H \trianglelefteq G$.

Proof: Suppose $|G : H| = 2$. Let $a \in G - H$ so the left-cosets of H are precisely H and aH. Now, it is necessarily the case that $aH = G - H$, because there are just two cosets, and any element not in H must be in the other coset aH, and conversely.

Similarly, for right cosets we have $Ha = G - H = aH$. This establishes $Ha = aH$, or rather $H = aHa^{-1}$ for all $a \in G - H$. On the other hand, if $a \notin G - H$, then $a \in H$ and $H = aHa^{-1}$ trivially.

We've now reasoned that $H = aHa^{-1}$ for all $a \in G$. Thus $H \trianglelefteq G$. ■

6. Prove that characteristic subgroups are normal.

Proof: Suppose H is characteristic in G.
This means that $\varphi(H) = H$ for any $\varphi \in \mathrm{Aut}(G)$.
Given $g \in G$, let $\varphi_g \in \mathrm{Aut}(G)$ be the inner automorphism $\varphi_g(x) = gxg^{-1}$.
Then $\varphi_g(H) = H$, which means $gHg^{-1} = H$.
It follows that H is normal. ■

7. Prove that a group of order 56 has a normal Sylow p-group for some prime p dividing its order.

Proof: As $56 = 2^3 \cdot 7$, the only primes dividing its order are 2 and 7. Thus we seek a normal Sylow 2-subgroup P (of order $2^3 = 8$), or a normal Sylow 7-subgroup Q (of order 7).

Let $Q \in Syl_7(G)$. If $n_7 = 1$, then $Q \trianglelefteq G$, in which case we are done. Otherwise, assume $n_7 > 1$. Sylow's theorem asserts $n_7 = 1 + 7k$, for some integer k, and $n_7|8$. The only possibility is $n_7 = 8$.

Let the 8 Sylow 7-groups be $\{Q_1, Q_2, Q_3, \dots, Q_8\}$, with $Q_1 = Q$. If $i \neq j$, then $Q_i \cap Q_j$ is a proper subgroup of $Q_i \cong Z_7$, so $Q_i \cap Q_j = 1$. Thus the sets $Q_i - \{1\}$ are disjoint. Let $X = \bigcup_{i=1}^{8}(Q_i - \{1\})$, so $|X| = 8 \cdot 6 = 48$. Any element of X is a non-identity element of some $Q_i \cong Z_7$, and therefore has order 7. Note that G has exactly $56 - 48 = 8$ elements that are not in this union, and one of these elements is 1. Say $G - X = \{1, g_1, g_2, g_3, \dots g_7\}$.

Now, consider a Sylow 2-subgroup P, for which $|P| = 2^3 = 8$. As no element of P has order 7, it is necessarily the case that $P = \{1, g_1, g_2, g_3, \dots g_8\}$. This is the only possibility for P, so we conclude that P is the unique Sylow 2-subgroup, hence P is normal.

In conclusion, either Q or P is normal. ■