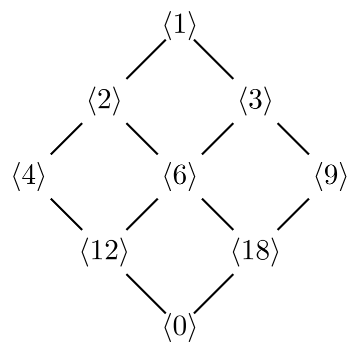
Abstract Algebra I MATH 601 Midterm October 29, 2015

Name:________________ R. Hammack Score:_______

1. Short Answer (8 points each)

(a) Draw the subgroup lattice for $\mathbb{Z}/36\mathbb{Z}$.



(b) List all Sylow subgroups of $\mathbb{Z}/36\mathbb{Z}$.

Because $36 = 2^2 3^2$, there is a Sylow 2-subgroup of order 4 and a Sylow 3-subgroup of order 9.
Sylow 2-subgroup: $\{0, 9, 18, 27\}$
Sylow 3-subgroup: $\{0, 4, 8, 12, 16, 20, 24, 28, 32\}$

Any other Sylow subgroup is conjugate to one of these. But these are normal because $\mathbb{Z}/36\mathbb{Z}$ is abelian. Thus these are conjugate only to themselves. Hence they are the only Sylow subgroups.

(c) Find a representative of each conjugacy class of elements of order 4 in S_8.

(1 2 3 4)
(1 2 3 4)(5 6)
(1 2 3 4)(5 6)(7 8)
(1 2 3 4)(5 6 7 8)

(d) State Cauchy's Theorem.

If a prime p divides the order of a finite group G, then G has an element of order p.

(e) Give an example a subgroup that is normal but not characteristic.

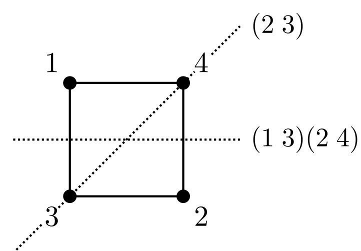
Let $G = Z_2 \times Z_2$. Note that that $H = Z_2 \times \{1\} \trianglelefteq G$ because G is abelian. However H is not characteristic because it is not fixed by the automorphism $\varphi \in \mathrm{Aut}(G)$ for which $\varphi(x, y) = (y, x)$.

2. Prove that $H \leq C_G(H)$ if and only if H is abelian.

Proof. Suppose $H \leq C_G(H) = \{g \in G \mid gh = hg \; \forall h \in H\}$. This means every $g \in H$ satisfies $gh = hg$ for all $h \in H$. Consequently H is abelian.

Conversely suppose H is abelian. Take any $g \in H$. We claim $g \in C_G(H)$. Because H is abelian it follows that $gh = hg \; \forall h \in H$. This means $g \in C_G(H) = \{g \in G \mid gh = hg \; \forall h \in H\}$. Thus $H \leq C_G(H)$.

3. Prove that the subgroup of S_4 generated by $(1\ 2)$ and $(1\ 3)(2\ 4)$ is isomorphic to D_8.



Draw a square with vertices labeled as above. Any symmetry of the square (i.e. any element of D_8) corresponds to a permutation of corners $\{1, 2, 3, 4\}$. In this way we regard D_8 as a subgroup of S_4

Reflection s across the x-axis is identified with the permutation $s = (1\ 3)(2\ 4)$. Likewise reflection across the diagonal line through 3 and 4 is identified with the permutation $(1\ 2)$. Consequently the subgroup of S_4 generated by $s = (1\ 3)(2\ 4)$ and $(1\ 2)$ is isomorphic to a subgroup of D_8. We will argue that it is all of D_8.

Now, rotation by $90°$ counterclockwise corresponds to the permutation $(1\ 3\ 2\ 4) = (1\ 2) \cdot (1\ 3)(2\ 4)$. Then rotation by $90°$ clockwise is $r = (1\ 3\ 2\ 4)^{-1} = ((1\ 2) \cdot (1\ 3)(2\ 4))^{-1}$.

Consequently the subgroup of D_8 generated by $(1\ 2)$ and $(1\ 3)(2\ 4)$ contains both r and s. Because r and s generate all of D_8, we see that the subgroup of S_4 generated by $(1\ 2)$ and $(1\ 3)(2\ 4)$ is D_8.

4. Suppose $A \trianglelefteq G$, and A is abelian. Recall that in this situation $AB \leq G$. Let $B \leq G$ be any subgroup. Prove $A \cap B \trianglelefteq AB$.

Proof. First note that $A \cap B \subseteq AB$ because any $x \in A \cap B$ satisfies $x \in A$ and therefore $x = x1 \in AB$. We also know $A \cap B$ is a subgroup of G because it is the intersection of two subgroups. It follows that $A \cap B \leq AB$. We now need to show that it is a *normal* subgroup of AB.

Take elements $g \in A \cap B$ and $ab \in AB$. We must argue that $(ab)g(ab)^{-1} \in A \cap B$.

First we will show that $(ab)g(ab)^{-1} \in A$. Note $(ab)g(ab)^{-1} = a(bgb^{-1})a^{-1}$. As $g \in A$ and $A \trianglelefteq G$, it follows that $bgb^{-1} \in A$. Then $a(bgb^{-1})a^{-1} \in A$ because it is a product of elements of A. Consequently $(ab)g(ab)^{-1} \in A$.

Next we will show that $(ab)g(ab)^{-1} \in B$. In the previous paragraph we showed $(ab)g(ab)^{-1} = a(bgb^{-1})a^{-1}$, with $bgb^{-1} \in A$. But also, $a, a^{-1} \in A$, and A is abelian, so $(ab)g(ab)^{-1} = a(bgb^{-1})a^{-1} = aa^{-1}(bgb^{-1}) = bgb^{-1}$. But $g, b \in B$, so $bgb^{-1} \in B$. Thus we have established $(ab)g(ab)^{-1} \in B$.

By the previous two paragraphs, $(ab)g(ab)^{-1} \in A \cap B$, so $A \cap B \trianglelefteq AB$

5. Suppose G is a group of odd order. Prove that for any non-identity element $x \in G$, x and x^{-1} are not conjugate in G.

Proof. Suppose for the sake of contradiction that G has odd order and there are elements $x, g \in G$ with $x \neq 1$ and $gxg^{-1} = x^{-1}$. (That is, x is conjugate to x^{-1}.)

Now, if it happened that $x = x^{-1}$, then $x^2 = 1$. and we would have a subgroup $\langle x \rangle \leq G$ of order 2. By Lagrange's Theorem 2 divides the odd number $|G|$, which is a contradiction.

Thus for the remainder of the proof we will assume $x \neq x^{-1}$. We now show the order of g is even. Taking the inverse of $gxg^{-1} = x^{-1}$ yields $gx^{-1}g^{-1} = x$. From these we get

$$g^2xg^{-2} = g\left(gxg^{-1}\right)g^{-1} = gx^{-1}g^{-1} = x.$$

And once again

$$g^4xg^{-4} = g^2\left(g^2xg^{-2}\right)g^{-2} = g^2xg^{-2} = x,$$

and so on, so that for any positive m we have

$$g^{2m}xg^{-2m} = g^2\left(g^{2m-2}x\,g^{-2m+2}\right)g^{-2} = g^2xg^{-2} = x.$$

From this

$$g^{2m+1}xg^{-2m-1} = g\left(g^{2m}x\,g^{-2m}\right)g^{-1} = gxg^{-1} = x^{-1}.$$

Thus we have determined that $g^{2m+1}xg^{-(2m+1)} = x^{-1} \neq x$, and it follows that $g^{2m+1} \neq 1$ for any m. Therefore the order of g is even. But the order of g must divide the odd number $|G|$, a contradiction.

6. Prove that $Z(S_n) = 1$ for all $n \geq 3$.

   Suppose $n \geq 3$ and let $\pi \in S_n$ be any non-identity permutation. We will show that $\pi \notin Z(S_n)$ by producing a $\mu \in S_n$ for which $\pi\mu \neq \mu\pi$.

   Since $\pi \neq 1$ there are distinct elements $a, b \in \{1, 2, 3, \ldots n\}$ for which $\pi(a) = b$. Select $\mu \in S_n$ for which $\mu(a) = b$, but $\mu(b) \neq \pi(b)$. Note that this is always possible. If $\pi(b) = c \notin \{a, b\}$ then we can let μ be the transposition of a and b. On the other hand, if $\pi(b) = a$ we can find a third element $c \in \{1, 2, \ldots, n\}$ and make $\mu(b) = c$.

$$\begin{tikzcd} a \arrow[r, "\pi"] \arrow[r, dotted, shift right] & b \arrow[r, "\pi"] \arrow[l, dotted, bend left, "\mu"] & c \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r] \arrow[r, dotted, shift right, "\mu"'] & b \arrow[l, bend right, "\pi"'] \arrow[r, dotted, "\mu"'] & c \end{tikzcd}$$

   Notice that $\pi\mu(a) = \pi(\mu(a)) = \pi(b) \neq \mu(b) = \mu(\pi(a)) = \mu\pi(a)$. This means $\pi\mu \neq \mu\pi$, so π does not commute with everything in S_n and is therefore not in its center.

   Because no non-identity permutation is in $Z(S_n)$ it follows that $Z(S_n) = 1$.

7. Let G be a group of order 200. Prove that G has a normal Sylow 5-subgroup.

   Proof. Note that $200 = 2^2 5^2$.
   By Sylow's Theorem, G has $n_5 \geq 1$ Sylow 5-subgroups.
   Also by Sylow's theorem n_5 divides $2^3 = 8$, and $n_5 \equiv 1 \pmod 5$.
   Thus $n_5 \mid 8$ and $n_5 \in \{1, 6, 11, 16, \ldots\}$.
   The only possibility is $n_5 = 1$. Thus there is only one Sylow 5-subgroup; call it P.
   As all Sylow 5-subgroups are conjugate, we conclude $gPg^{-1} = P$ for any $g \in G$.
   We have produced a normal Sylow 5-subgroup.