Abstract Algebra I — **Final Exam** — December 13, 2011
MATH 601 — R. Hammack

Directions: Solve five of the following ten questions.

A. Groups

1. Recall that if $H \leq G$, then $[G, H]$ is the subgroup of G generated by all commutators $[g, h] = g^{-1}h^{-1}gh$ with $g \in G$ and $h \in H$.

Prove that H is normal in G if and only if $[G, H] \leq H$.

2. Suppose A and B are normal subgroups of G such that G/A and G/B are both abelian. Prove that $G/(A \cap B)$ is abelian.

3. The matrix $\begin{bmatrix} 0 & -1 \\ 1 & 4 \end{bmatrix}$ has order 5 in $\mathrm{GL}_2(\mathbb{F}_{19})$. Use it to construct a non-abelian group of order 1805.

B. Rings

4. Let R be a ring with 1. Arguing strictly from the definition of a ring, show that $(-1)^2 = 1$.

5. Let R be a ring with 1. Prove that the center of the ring $M_2(R)$ is $Z = \left\{ \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \mid r \in Z(R) \right\}$.

6. The *characteristic* of a ring R is the smallest positive integer p such that $1 + 1 + 1 + \cdots + 1 = 0$ (p times) in R. If no such integer exists, then we say R has *characteristic* 0.

Prove that if an integral domain has characteristic p, then p is either prime or zero.

7. Recall that an element a in a ring R is *nilpotent* if $a^n = 0$ for some integer n. Let R be a commutative ring with $1 \neq 0$.

Prove that if a is nilpotent, then $1 - ab$ is a unit for every $b \in R$.

8. Suppose R and S are rings with identities. Prove that every ideal of $R \times S$ is of form $I \times J$, where I is an ideal in R and J is an ideal in S.

9. Prove that the quotient ring $\mathbb{Z}[i]/I$ is finite for every non-zero ideal I in the Gaussian integers $\mathbb{Z}[i]$.

10. Prove that the quotient of a PID by a prime ideal is again a PID.