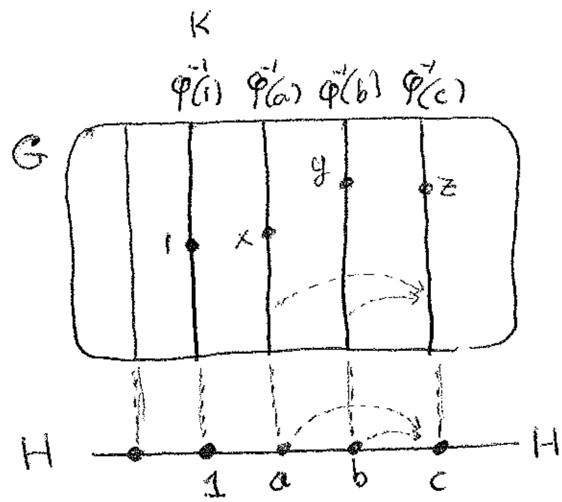
Chapter 3 Quotient Groups and Homomorphisms

Section 3.1 Definitions and Examples

Basic setup: Consider surjective homomorphism $\varphi : G \longrightarrow H$.

Fiber: Given $a \in H$, fiber over a is $\varphi^{-1}(a) = \{x \in G \mid \varphi(x) = a\} \subseteq G$

Kernel: The kernel of φ is $K = \varphi^{-1}(1) = \{x \in G \mid \varphi(x) = 1\} \leq G$



$\varphi(x) = a$
$\varphi(y) = b$
$\varphi(z) = c$, etc.

The set of fibers forms a group that is $\cong$ to H.

Mult: $\varphi^{-1}(a)\varphi^{-1}(b) = \varphi^{-1}(ab)$ (mimics H)

Can prove: If $\varphi(x) = a$, then

$\varphi^{-1}(a) = xK = \{xk \mid k \in K\}$ ← "left coset of K"

$\varphi^{-1}(a) = Kx = \{kx \mid k \in K\}$ ← "right coset of K"

Theorem 3 Multiplication of fibers works like this:

$$\begin{matrix} \varphi^{-1}(a) & \varphi^{-1}(b) & = & \varphi^{-1}(ab) \\ \| & \| & & \| \\ xK & \cdot\, yK & = & xyK \end{matrix}$$

Definition Given a homomorphism $\varphi : G \to H$ (not necessarily surjective) with kernel K, the factor group or quotient group is

$$G/K = \{\varphi^{-1}(a) \mid a \in H\} = \text{set of fibers}$$
$$= \{xK \mid x \in G\} = \text{set of cosets}$$

with operation defined as in Theorem 3.

Before an example, some loose ends:

- **Cosets** can be defined for *any* subgroup $K \leq G$, whether or not it's a kernel.

$$\left.\begin{array}{l} xK = \{xk \mid k \in K\} \\ Kx = \{kx \mid k \in K\} \end{array}\right\} \begin{array}{l} \text{In general, } xK = Kx \;\; \forall x \in G \\ \iff K \text{ is kernel of some homom} \end{array}$$

- If G has operation $+$ ($\therefore$ abelian) then cosets are

$$x+K = \{x+k \mid k \in K\} \quad \text{and} \quad x+K = K+x.$$

- Left cosets of K form a partition of G.
- Right " " " " " " " "
- $xK = yK \iff x^{-1}y \in K \iff y = xk$ for some $k \in K$
- $Kx = Ky \iff xy^{-1} \in K \iff y = kx$ " " "
- $x+K = y+K \iff x-y \in K \iff y = x+k$ " " "

Example 1 Consider homomorphism $\varphi: \mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ defined as $\varphi(m) = \overline{m}$. $\left(\varphi(m+n) = \overline{m+n} = \overline{m} + \overline{n} = \varphi(m) + \varphi(n) \right)$

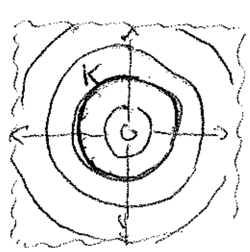
K :	$1+K$	$2+K$	$3+K$
-4	-3	-2	-1
0	1	2	3
4	5	6	7
8	9	10	11
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$\overline{0} \quad \overline{1} \quad \overline{2} \quad \overline{3}$ $\quad \mathbb{Z}/4\mathbb{Z}$

Note $\mathbb{Z}/4\mathbb{Z}$ was earlier defined as set of equiv classes mod 4.

These are fibers of φ, and operation matches definition of quotient groups.

Example 2 Consider homomorphism $\varphi: \mathbb{C}^\times \to \mathbb{R}^+$, $\varphi(z) = |z|$



Kernel is $\{z \in \mathbb{C} \mid \varphi(z) = 1\} = \{z \in \mathbb{C} \mid |z| = 1\} =$ unit circle

zK = circle of radius $|z|$ $= \varphi^{-1}(|z|)$ = fiber over $|z|$

These circles form the group $\mathbb{C}^\times/K \cong \mathbb{R}^+$

$\varphi: \mathbb{C}^\times \longrightarrow \mathbb{R}^+$

φ "closes the umbrella"

Normal Subgroups

Definition A subgroup $N \leq G$ is <u>normal</u> (expressed $N \trianglelefteq G$) if $gN = Ng$ for all $g \in G$. (Text. $gNg^{-1} = N$)

Proposition 5 $N \trianglelefteq G \iff$ (Left cosets $\{gN \mid g \in G\}$ form a group with operation $gN \cdot hN = ghN$)

Definition If $N \trianglelefteq G$, then $G/N = \{gN \mid g \in G\}$ with above operation

Theorem 6 Suppose $N \leq G$. The following are equivalent:

1. $N \trianglelefteq G$
2. $N_G(N) = G$ (Recall $N_G(N) = \{g \in G \mid gxg^{-1} \in N \ \forall x \in N\}$)
3. $gN = Ng$ $gNg^{-1} = N$ {Here: $gNg^{-1} = \{gng^{-1} \mid n \in N\}$}
4. Left cosets form a group
5. $gNg^{-1} \subseteq N$

Proposition 7 $N \trianglelefteq G \iff$ (N is the kernel of some homomorphism $\pi : G \to H$.)

Proof ($\Leftarrow$) By earlier discussion today, if N is a kernel, the $xH = Hx$, so $N \trianglelefteq G$.

($\Rightarrow$) Suppose $N \trianglelefteq G$. Define

$$\pi : G \longrightarrow G/N$$
$$\pi(g) = gN$$

Check this is a homomorphism with kernel N.

π is called the "natural projection homomorphism.