Section 2.4 Subgroups Generated by Subsets.

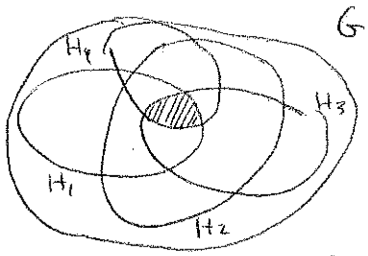
Goal Answer the following question:

We know $\langle a \rangle = \{ a^n \mid n \in \mathbb{Z} \} = \langle \{a\} \rangle \leq G$

What is $\langle A \rangle$, where $A \subseteq G$?

Proposition If $H \leq G$ and $K \leq G$ then $H \cap K \leq G$.

Proposition If $\{ H_\alpha \mid \alpha \in I \}$ is a collection of subgroups $H_\alpha \leq G$ for each $\alpha \in I$, then $\bigcap_{\alpha \in I} H_\alpha \leq G$



$I = \{1, 2, 3, 4\}$

$\bigcap_{\alpha \in I} H_\alpha = H_1 \cap H_2 \cap H_3 \cap H_4 \leq G$

But keep in mind index set I could have any cardinality.

Definition Given $A \subseteq G$, $\langle A \rangle = \bigcap_{\substack{A \subseteq H \\ H \leq G}} H \leq G$

Example $G = \mathbb{Z}$, $A = \{72, 36, 24\}$

$\langle A \rangle = 2\mathbb{Z} \cap 4\mathbb{Z} \cap 3\mathbb{Z} \cap 6\mathbb{Z} \cap 12\mathbb{Z} \cap \mathbb{Z}$
$= 12\mathbb{Z}$

Example $G = \mathbb{R}^\times$ $A = \{ \pi, \sqrt{2}, 7, e, \sqrt{11} \}$

$\langle A \rangle = ???$

Even though the definition defines $\langle A \rangle$ unambiguously, it's hard to say exactly what $\langle A \rangle$ is.

<u>Definition</u> If $A \subseteq G$, then

$$\bar{A} = \{ a_1^{\varepsilon_1} a_2^{\varepsilon_2} a_3^{\varepsilon_3} \cdots a_n^{\varepsilon_n} \mid a_i \in A \quad \varepsilon_i = \pm 1 \}$$
$$= \{ a_1^{p_1} a_2^{p_2} a_3^{p_3} \cdots a_n^{p_n} \mid a_i \in A \quad p_i \in \mathbb{Z} \}$$
$$= \text{finite products of powers of elements of } A.$$

<u>Proposition 9</u> $\bar{A} = \langle A \rangle$

$\uparrow$ of theoretical use (proofs) $\qquad \uparrow$ of computational use

It's good to have two different ways of looking at the same thing.

<u>Note</u>: If G is abelian, and $A = \{a_1, a_2, \dots, a_k\}$, then
$$\bar{A} = \{ a_1^{p_1} a_2^{p_2} \cdots a_k^{p_k} \mid k \geq 0,\ p_i \in \mathbb{Z} \} \quad \text{(in order)}$$

<u>Convention</u>: $\langle \phi \rangle = \bar{\phi} = \{1\}$

<u>Notation</u> $\langle \{a_1, a_2, \dots a_n\} \rangle = \langle a_1\ a_2 \dots a_n \rangle$

$$\langle A \cup B \rangle = \langle A, B \rangle$$

<u>Example</u> $\langle \pi, \sqrt{2}, 7, e\ \sqrt{\pi} \rangle$
$$= \{ x\pi + y\sqrt{2} + z7 + ue + w\sqrt{\pi} \mid x, y, z, u, w \in \mathbb{R} \} \leq \mathbb{R}$$

(proper subgroup of $\mathbb{R}$, <u>not</u> cyclic)

Section 7.5 Subgroup Lattices

Example Consider $\mathbb{Z}/36\mathbb{Z}$.

Divisors of 36: 1, 2, 3, 4, 6, 9, 12, 18, 36

Subgroups of $\mathbb{Z}/36\mathbb{Z}$:

$\langle 1 \rangle = \{0, 1, 2, \dots 35\} = \mathbb{Z}/36\mathbb{Z}$

$\langle 2 \rangle = \{0, 2, 4, \dots 34\}$

$\langle 3 \rangle = \{0, 3, 6, \dots 33\}$

$\langle 4 \rangle = \{0, 4, 8, 12 \dots 32\}$

$\langle 6 \rangle = \{0, 6, 12, 18, 24, 30\}$

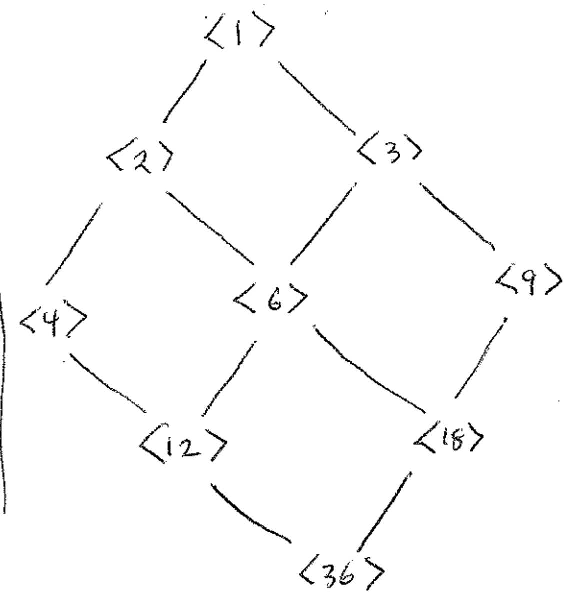
$\langle 9 \rangle = \{0, 9, 18, 27\}$

$\langle 12 \rangle = \{0, 12, 24\}$

$\langle 18 \rangle = \{0, 18\}$

$\langle 36 \rangle = \{0\}$

Subgroup lattice of $\mathbb{Z}/36\mathbb{Z}$



Example Consider Q_8

Subgroups of Q_8:

Q_8

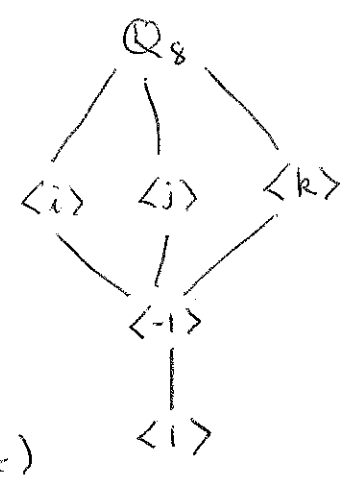
$\langle i \rangle = \langle i\ i^2, i^3\ i^4 \rangle = \{i\ -1, -i\ 1\}$

$\langle j \rangle = \{j, -1, -j\ 1\}$

$\langle k \rangle = \{k, -1, -k, 1\}$

$\langle -1 \rangle = \{1, -1\}$

$\langle 1 \rangle = \{1\}$ (Note $\langle i, j \rangle = Q_8$, etc)



Text makes the point that subgroup lattices can help in the computation of centralizers and centers.

$C_{Q_8}(\{i\}) = \langle i \rangle$ because this is smallest subgroup containing i that commutes with everything in $\{i\}$.

Also $Z(Q_8) = \langle -1 \rangle = \{1, -1\}$.