Section 1.7 Continued

Example of an Action

$G = GL_2(\mathbb{Z}/2\mathbb{Z})$ acts on $A = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

(entries of vectors are from $\mathbb{Z}/2\mathbb{Z}$)

The action is matrix multiplication.

Thus there is a homomorphism

$\varphi : G \longrightarrow S_A$ where $\varphi(B) = \sigma_B$.

This map is injective:

Suppose $B, B' \in G$ and $\varphi(B) = \varphi(B')$

$$\sigma_B = \sigma_{B'}$$

$$\sigma_B(x) = \sigma_{B'}(x) \quad \forall\, x \in A$$

$$Bx = B'x \quad \forall\, x \in A$$

$$\left.\begin{array}{l} \underbrace{B\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\text{1st col of } B} = \underbrace{B'\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\text{1st col of } B'} \\ \\ \underbrace{B\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\text{2nd col of } B} = \underbrace{B'\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\text{2nd col of } B'} \end{array}\right\} \text{Therefore } B = B'$$

Have shown $\varphi(B) = \varphi(B') \Rightarrow B = B'$ so φ injective.

Therefore action is <u>faithful</u>.

<u>Note</u> Injection $\varphi : G \to S_A$ and $|G| = |S_A| = 6$

so φ is also <u>surjective</u>.

We have a bijective homomorphism $\varphi : GL_2(\mathbb{Z}/2\mathbb{Z}) \to S_A$

so $GL_2(\mathbb{Z}/2\mathbb{Z}) \cong S_A \cong S_3$

Chapter 2 Subgroups (Sections 2.1, 2.2)

Definition A subgroup of a group G is a subset $H \subseteq G$ that is itself a group under G's operation. We write $H \leq G$ to indicate H is a subgroup of G.

Subgroup Criteria: Suppose G is a group, $H \subseteq G$.

Proposition $H \leq G$ if and only if

① $e \in H$ {or $H \neq \phi$}

② $x, y \in H \Rightarrow xy \in H$ (H is closed)

③ $x \in H \Rightarrow x^{-1} \in H$ (H is closed under inverses)

Proposition $H \leq G$ if and only if

① $H \neq \phi$.

② $\forall\, x, y \in H$, $xy^{-1} \in H$.

Proposition Suppose H is finite. Then $H \leq G$ if and only if

① $H \neq \phi$

② $x, y \in H \Rightarrow xy \in H$

Reason: Suppose $H \neq \phi$ finite and closed under mult.

Take $x \in H$. Look at $\{x\ x^2\ x^3 \ldots\} \subseteq H$

Must have $x^p = x^q$ for some $p < q$.

Then $e = x^{q-p} \in H$

So $e = (x)(x^{q-p-1})$

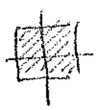
Thus $x^{-1} = x^{q-p-1} \in H$.

Examples (Kernels and Stabilizers of actions)

Suppose G acts on a set A.

The kernel of the action is
$\{ g \in G \mid ga = a \ \ \forall\ a \in G \} \leq G$ (This is a subgroup)

Given $s \in A$, the stabilizer of s is
$\{ g \in G \mid g.s = s \} \leq G$ (This is a subgroup)

Example
$\mathbb{Z}$ acts on $A =$  ga = rotation of a by $g \cdot 90°$ clockwise.

Kernel: $\{ g \in \mathbb{Z} \mid ga = a \} = \{ 0, \pm 4, \pm 8, \ldots \} = 4\mathbb{Z} \leq \mathbb{Z}$

Stabilizers of $s = (1,1)$ $\{ g \in \mathbb{Z} \mid g.s = s \ \ = 4\mathbb{Z}$
Stabilizer of $s = (0,0)$ $\{ g \in \mathbb{Z} \mid g.s = s \} = \mathbb{Z}$

$\mathbb{Z}$ also acts on $\mathcal{P}(A)$. Given $X \in \mathcal{P}(A)$, i.e. $X \subseteq A$ we put $g.X = \{ g.x \mid x \in X \}$. (Check that this is an action).

Stabilizer of $X = \{ (1,1), (-1,-1) \}$ 
is $\{ g \in \mathbb{Z} \mid gX = X \} = 2\mathbb{Z} \leq \mathbb{Z}$.

Example G acts on itself by conjugation.
Given $g \in G$ and $a \in G$, $g.a = gag^{-1}$.
(Check that this is an action.)

Kernel of action is
$\{ g \in G \mid g.a = a \} = \{ g \in G \mid gag^{-1} = a \ \ \forall\ a \in G \}$
$= \{ g \in G \mid ga = ag \ \ \forall\ a \in G \} \leq G$

This is the set of all $g \in G$ that commute with everything in G. Called the center of G.

Centers Centralizers and Normalizers

The center of a group G is the subgroup

$$Z(G) = \{ g \in G \mid gx = xg \quad \forall\, x \in G \} \leq G$$

{elements of G That commute with everything}

If $A \subseteq G$, the centralizer of A is

$$C_G(A) = \{ g \in G \mid gx = xg \quad \forall\, x \in A \} \leq G$$

{elements of G That commute with everything in A}

Thus $C_G(G) = Z(G) \qquad C_G(\{e\}) = G$

Exercise: Show $Z(GL_n(\mathbb{F})) = \left\{ \begin{bmatrix} d & 0 & \cdots & 0 \\ 0 & d & & \vdots \\ 0 & \cdots & \ddots & d \end{bmatrix} \;\middle|\; d \in \mathbb{F} \right\}$

By Homework § 1.2, 4

$$Z(D_{2(2k)}) = \{ 1, r^k \}$$

$$C_G(A) = \{ g \in G \mid gxg^{-1} = x \quad \forall\, x \in A \} \leq G$$

The normalizer of A in G is the subgroup

$$N_G(A) = \{ g \in G \mid gxg^{-1} \in A \quad \forall\, x \in A \} = \{ g \in G \mid gAg^{-1} = A \}$$

Note $Z(G) \leq C_G(A) \leq N_G(A)$