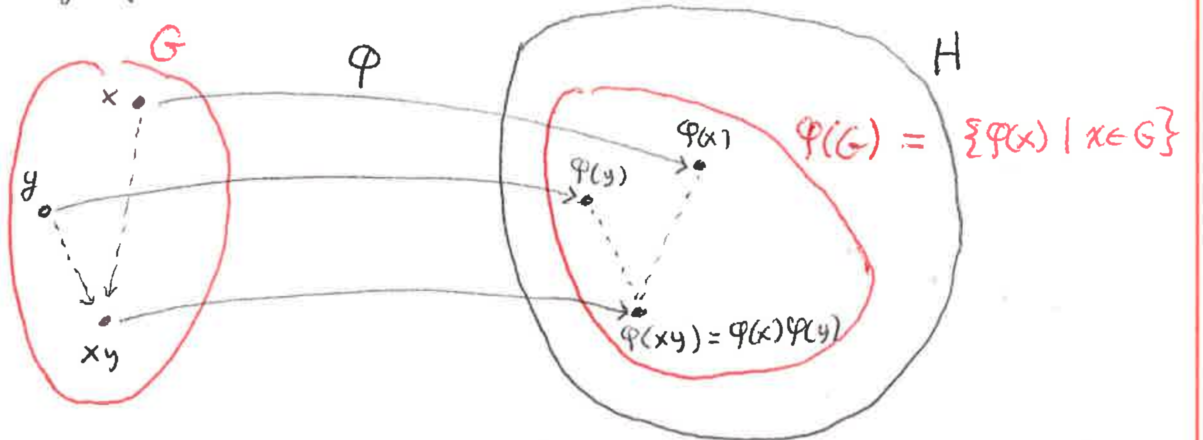
Homework

1.6 4, 5, 6, 9, 13, 18

1.7 6, 8

Section 1.6 Homomorphisms and Isomorphisms

Given groups G and H, there are lots of maps $\varphi : G \to H$. From an algebraic point of view, we are mainly interested in those maps that respect the group structure of G and H in the sense that $\varphi(xy) = \varphi(x)\varphi(y)$



Definition A homomorphism $\varphi : G \to H$ is a map for which $\varphi(xy) = \varphi(x)\varphi(y)$ $\forall\, x, y \in G$.

Ex $\varphi : GL_n(\mathbb{R}) \to \mathbb{R}^\times = \mathbb{R} - \{0\}$

$\varphi(A) = \det(A)$

$\varphi(AB) = \det(AB) = \det(A)\det(B) = \varphi(A)\varphi(B)$.

Ex $\varphi : \mathbb{Z} \to \mathbb{R}^\times$ $\quad \varphi(k) = 2^k$

$\varphi(x+y) = 2^{x+y} = 2^x 2^y = \varphi(x)\varphi(y)$

Ex $\varphi : \mathbb{R}^\times \to \{1, -1\}$ $\quad \varphi(x) = \frac{x}{|x|}$

$\varphi(xy) = \frac{xy}{|xy|} = \frac{xy}{|x||y|} = \frac{x}{|x|}\frac{y}{|y|} = \varphi(x)\varphi(y)$

Ex $\varphi : \mathbb{R}^\times \to \{1, -1\}$ $\quad \varphi(x) = 1$

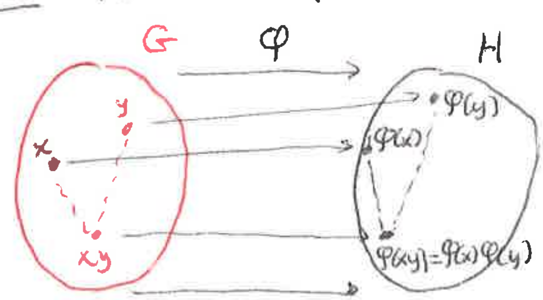
Note $\varphi(a^n) = \varphi(a a \cdots a) = \varphi(a)\varphi(a)\cdots\varphi(a) = \varphi(a)^n$ for any homo.

$\varphi(1_G) = 1_H$

$\varphi(1)\varphi(a) = \varphi(1 a) = \varphi(a)$

Thus $\varphi(1_G) = 1_H$

Definition An isomorphism $\varphi : G \to H$ is a bijective homomorphism



If there is an iso $\varphi : G \to H$, we say $G \cong H$.

φ "lays G on top of H" so that they match. H is the same group as G, with elements x relabled as $\varphi(x)$.

If $G \cong H$, any algebraic structure that one group has is shared by the other.

<u>Examples</u> $\mathbb{R}^+ \cong \mathbb{R}$ because $\ln : \mathbb{R}^+ \to \mathbb{R}$ is iso.

$$\ln(xy) = \ln(x) + \ln(y)$$

$\mathbb{R}^\times \not\cong \mathbb{R}$ because $\mathbb{R}^\times$ has element -1 of order 2, but $\mathbb{R}$ has no such.

Note, however, if $\varphi : G \to H$ is just a homomorphism, then G and H can have different structures.

Example $\det : GL_2(\mathbb{R}) \longrightarrow \mathbb{R}^\times$

$GL_2(\mathbb{R})$: non-abelian; $\mathbb{R}^\times$: abelian

<u>Defining Homomorphisms between groups with presentations.</u>

<u>Example</u>

$$G = \langle a, b \mid a^6 = 1,\ b^4 = 1 \rangle = \{ a^{k_1} b^{l_1} a^{k_2} b^{l_2} \cdots \mid 0 \le k_i < 6,\ 0 \le l_i < 4 \}$$

$$H = \langle x, y \mid x^3 = 1,\ y^2 = 1,\ xy = yx \rangle \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

Here any relation that holds for $a, b \in G$ also hold in H when x, y are replaced by a, b, respectively

There is a unique homomorphism $\varphi : G \to H$ for which $\varphi(a) = x$, $\varphi(b) = y$.

~~It is~~ $\cancel{\varphi(a^{k_1} b^{l_1} a^{k_2} b^{l_2} \cdots) = x^{k_1} y^{l_1} x^{k_2} y^{l_2} \cdots = \varphi(a)^{k_1} \varphi(b)^{l_1} \varphi(a)^{k_2} \cdots}$

Define $\varphi(a^k) = x^k = \varphi(a)^k$ $\qquad 1 = \varphi(a^6) = x^6 = (x^3)^2 = 1^2 = 1$

$\varphi(b^l) = y^l = \varphi(b)^l$

For this to work, we need relation $a^6 = 1$ to hold in H as $x^6 = 1$

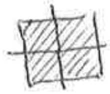
In general, define

$$\varphi(a^{k_1} b^{l_1} a^{k_2} b^{l_2} \cdots) = \varphi(a)^{k_1} \varphi(b)^{l_1} \varphi(a)^{k_2} \varphi(b)^{l_2} \cdots$$

Note that this is a homomorphism.

Section 1.7 Group Actions

Recall A permutation of a set A is a bijection $\pi : A \to A$.

Example $A =$  = set of points on plane.

$\{\pi, \mu \in S_A\}$

$\pi : A \to A$ is rotation by 90°
$\mu : A \to A$ is reflection over x-axis
} permutations of A.

Notation Let G be a group, A a set. Given $\varphi : G \times A \to A$ we write $\varphi(g,a) = g.a$

Definition An action of a group G on a set A is a map $G \times A \to A$ satisfying:

① $g'.(g.a) = (g'g).a \quad \forall g, g' \in G$ and $a \in A$

② $e.a = a \quad \forall a \in A$

Example $G = \mathbb{Z}$ $\quad A =$  $\quad \varphi(i,a) = i.a =$ rotation by $i\,90^\circ$

e.g. $0.a =$ rotation by $0 \cdot 90^\circ = 0^\circ$
$1.a =$ rotation by $1 \cdot 90^\circ = 90^\circ$
$2.a =$ " " " $2 \cdot 90^\circ = 180^\circ$ etc.

This is a group action

① $i.(j.a) = \begin{pmatrix}\text{rotation of } a \\ \text{by } j \cdot 90^\circ \text{ followed} \\ \text{by } i \cdot 90^\circ\end{pmatrix} = \begin{pmatrix}\text{rotation} \\ \text{by } i \cdot 90 + j \cdot 90 \\ = (i+j).90\end{pmatrix} = (i+j).a$

② $0.a = \begin{pmatrix}\text{rotation of } a \\ \text{by } 0 \cdot 90^\circ = 0^\circ\end{pmatrix} = a$

Definition If G acts on A, the kernel of this action is the set $\{g \in G \mid g.a = a \quad \forall a \in A\}$

In the example above, kernel $= \{0, \pm 4, \pm 8, \pm 12 \dots\} \subseteq \mathbb{Z}$.

FACTS (Proved in text) Given action of G on A:

If $g \in G$, ~~the func~~ $\sigma_g : A \to A$ ~~defined~~ as $\sigma_g(a) = g.a$
~~is a per~~

(i) For each $g \in G$, σ_g is a permutation of A.

$\{$ i.e. $\sigma_g : A \to A$ is bijection, i.e. $\sigma_g \in S_A \}$

(ii) The map $G \to S_A$ defined as $g \mapsto \sigma_g$ is a homomorphism

$\mu : G \to S_A \qquad \mu(g) = \sigma_g$

The action of G on A is faithful if

$G \longrightarrow S_A$ $\quad g \longmapsto \sigma_g$ is injective

(i.e. if distinct elements of G give distinct permutations of A)

$\boxed{\text{In such a case, a copy of } G \text{ "Lives" in } S_A}$ ← "faithful" copy of G in S_A

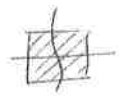
The example above ($\mathbb{Z}$ acting on square) is not faithful. $0, 4 \in \mathbb{Z}$ induce following permutations of A:

$$0 \longmapsto \sigma_0 = \text{rotation by } 0\cdot 90^\circ = 0^\circ$$

$$4 \longmapsto \sigma_4 = \text{rotation by } 4\cdot 90 = 360^\circ = \text{rotation by } 0^\circ.$$

Can you think of a faithful action on the square?

One answer:

$\mathbb{Z}/4\mathbb{Z}$ acts on $A =$ 

$g.a$ = rotation by $g\cdot 90^\circ$.