Section 1.3 Symmetric Groups

<u>Definition</u> Given set Ω, the <u>symmetric group on Ω</u> is
$S_\Omega = \{ f \mid f : \Omega \to \Omega \text{ is a bijection} \} = \left\{ \begin{array}{c} \text{set of all} \\ \text{bijections} \\ \Omega \to \Omega \end{array} \right\} = \left\{ \begin{array}{c} \text{permutations} \\ \text{of } \Omega \end{array} \right\}$

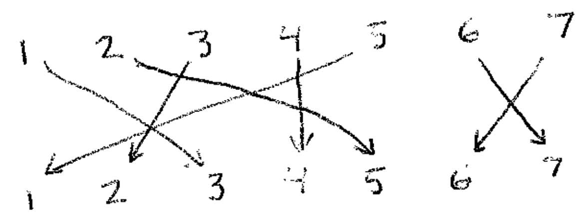
S_Ω is a group. Operation is composition

(i) composition is associative
(ii) Identity: $1 : \Omega \to \Omega$, $1(x) = x$
(iii) For $f \in S_\Omega$, $f^{-1} \in S_\Omega$ and $f \circ f^{-1} = f^{-1} \circ f = 1$.

If $\Omega = \{1, 2, 3, \ldots, n\}$, we write $S_\Omega = S_n =$
"symmetric group of degree n.

Typical element of S_7:

σ is a permutation of $\{1, 2, \ldots 7\}$

$\sigma =$ 

Structure of σ:

$1 \to 3 \to 2 \to 5$ 4 $6 \to 7$

σ decomposes into three smaller permutations called <u>cycles</u>
Notation: $\sigma = (1, 3\ 2\ 5)\ (4)\ (6\ 7)$

<u>Definition</u> A cycle $(a_1\ a_2\ a_3 \cdots a_k)$ is
a ~~the~~ element of S_n for which

$$\begin{array}{c} a_1 \to a_2 \\ a_2 \to a_3 \\ a_3 \to a_4 \\ \vdots \\ a_k \to a_1 \end{array}$$

and all other elements $\{1, 2, 3, \ldots n\} \sim \{a_1, a_2, \ldots a_k\}$ are sent to themselves

<u>Fact</u> Every permutation is a composition of disjoint cycles.

$$\sigma = (1, 3, 2\ 5)(4)(6\ 7)$$
$$= (1\ 3\ 2\ 5)(6\ 7)$$
$$= (3\ 2\ 5\ 1)(7\ 6)$$

<u>Fact</u> Disjoint cycles commute

$$(1\ 3\ 2\ 5)(6\ 7)$$
$$= (6\ 7)(1\ 3\ 2\ 5)$$

<u>Composition of cycles:</u>

$$(1\ 4\ 2)(1\ 3\ 2) = (1\ 3)(2\ 4)$$
$$(1\ 3\ 2)(1\ 4\ 2) = (1\ 4)(2\ 3)$$

<u>Inverse of a cycle</u>

$$(1\ 4\ 2)^{-1} = (2\ 4\ 1)$$

Check: $(1\ 4\ 2)(2\ 4\ 1) = (1)(2)(3)(4) = 1.$

$$\sigma = (1\ 3\ 2\ 5)(6\ 7)$$
$$\sigma^{-1} = (5\ 2\ 3\ 1)(7\ 6)$$

$$\sigma\sigma^{-1} = (1\ 3\ 2\ 5)(6\ 7)(5\ 2\ 3\ 1)(7\ 6) = (1)(2)(3)\ldots = 1$$

<u>Powers of cycles</u>

$$\pi = (1\ 2\ 3\ 4)$$
$$\pi^2 = (1\ 3)(4\ 1)$$
$$\pi^3 = (1\ 4\ 3\ 2)$$
$$\pi^4 = 1$$

a cycle of length k has order k

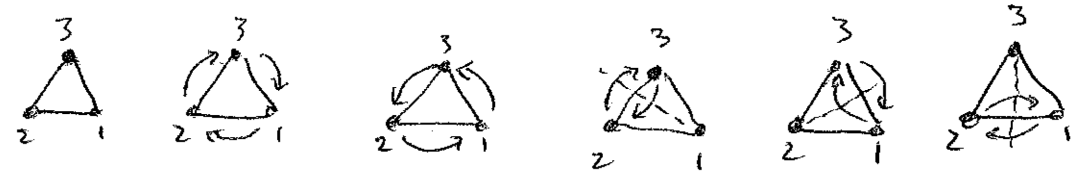
$$|(1\ 3\ 2\ 4)(5\ 6\ 7)(89)| = \operatorname{lcm}(4, 3, 2) = 12$$

<u>Examples</u>

S_2 = Permutations of $\{1,2\}$ $= \{1, (12)\}$

	1	(12)
1	1	(12)
(12)	(12)	1

$S_2 \cong \mathbb{Z}/2\mathbb{Z}$



$S_3 = \{1, (1\,2\,3), (1\,3\,2), (2\,3)\ (1\,3)\ (1\,2)\}$

$= \{1, r, r^2, \mu_1\ \mu_2\ \mu_3\}$

	1	r	r^2	μ_1	μ_2	μ_3
1	1	r	r^2	μ_1	μ_2	μ_3
r	r	r^2	1	μ_3	μ_1	μ_2
r^2	r^2	1	r	μ_2	μ_3	μ_1
μ_1	μ_1	μ_2	μ_3	1	r	r^2
μ_2	μ_2	μ_3	μ_1	r^2	1	r
μ_3	μ_3	μ_1	μ_2	r	r	1

S_3

$S_3 \cong D_6$

$S_4 \not\cong D_{\text{anything}}$