Section 5.4 Recognizing Direct Products.

Recall Suppose $H, K \leq G$. Then $HK = \{hk \mid h \in H, k \in K\}$

Proposition 14 (Ch. 3) $HK \leq G \iff HK = KH$

Corollary 15 (Ch. 3) $H \trianglelefteq G \implies HK \leq G$

Before getting to today's main task, a few words on commutators

Definition The commutator of $x, y \in G$ is $[x,y] = x^{-1}y^{-1}xy$

Note: $xy = yx\underbrace{[x,y]}_{\text{"fudge factor" measures non-commutativity of } x, y.}$

Note: G abelian $\iff [x,y] = 1 \quad \forall x, y \in G$

Rough Idea $\begin{pmatrix}\text{more}\\ \text{commutators}\end{pmatrix} = \begin{pmatrix}\text{more}\\ \text{non-abelian}\end{pmatrix}$

Definition Commutator subgroup of G is $G' = \underbrace{\langle [x,y] \mid x, y \in G\rangle}_{\text{complex non-abelian stuff}}$

Note (G abelian) $\iff G' = \{1\}$

Note G/G' is abelian $(xG')(yG') = xyG' = xy[y,x]G' = xyy^{-1}x^{-1}yxG' = yxG'$
$= (yG')(xG')$

Note G/G' is largest abelian quotient of G in the sense that if G/N is abelian, then $G' \leq N$

Reason: Suppose G/N is abelian. Then $xNyN = yNxN \Rightarrow$
$xyN = yxN \Rightarrow x^{-1}y^{-1}xyN = N \Rightarrow [x,y]N = N \Rightarrow [x,y] \in N$
$\Rightarrow \langle [x,y] \rangle \leq N \Rightarrow G' \leq N$.

Read the material on commutators. It will not be used much.

Decomposing Groups into Direct Products

{Main Question: Given G, is $G \cong H \times K$?}

Proof: Homework

Proposition 8 Let $H, K \leq G$. $HK = \{hk \mid h \in H, k \in K\}$
For $x \in HK$, the number of distinct ways to write $x = hk$ with $h \in H$, $k \in K$ is $|H \cap K|$

Proof (Homework)

Example
$G = Z_3 \times Z_3 \times Z_3$
$H = Z_3 \times Z_3 \times 1$
$K = 1 \times Z_3 \times Z_3$
$H \cap K = 1 \times Z_3 \times 1$

Thus $|H \cap K| = 3$

$(1,a,1) = \begin{cases} (1\,a\,1)(1\,1\,1) \\ (1\,1\,1)(1\,a\,1) \\ (1\,a^2\,1)(1\,a^2\,1) \end{cases}$

$(a,1,a) = \begin{cases} (a,1,1)(1,1,a) \\ (a\,a\,1)(1\,a^2\,a) \\ (a\,a^2\,1)(1\,a^2\,a) \end{cases}$

Theorem 9 Suppose:

① $H, K \trianglelefteq G$,

② $H \cap K = 1$.

Then $HK \cong H \times K$

In particular $\varphi: H \times K \longrightarrow HK$, $\varphi(h,k) = hk$ is an isomorphism.

Moreover $hk = kh$ $\forall$ $h \in H$ and $k \in K$.

As $(h,1)(1,k) = (1,k)(h,1)$.

[Read the proof. It's very instructive.]

$$\underset{HK}{\left(\begin{aligned} &hk \cdot h'k' \\ &= hh'kk' \end{aligned} \right)} \cong \underset{H \times K}{\left(\begin{aligned} &(h,k)(h',k') \\ &= (hh', kk') \end{aligned} \right)}$$

Tells when G is the direct product of its subgroups

$\boxed{\text{Consequence If } H,K \trianglelefteq G,\ H \cap K = 1 \text{ and } G = HK, \text{ then } G \cong H \times K.}$

Example $G = \mathbb{Z}_6 = \langle a \rangle = \{1, a, a^2, a^3, a^4, a^5\}$

$H = \langle a^2 \rangle = \{1, a^2, a^4\} \leq G$

$K = \langle a^3 \rangle = \{1, a^3\} \leq G$

Then $H, K \trianglelefteq G$, $G = HK$, $H \cap K = 1$, so $G = H \times K \cong \mathbb{Z}_3 \times \mathbb{Z}_2$

internal direct product

external direct product.

G	$\cong$ HK	$\cong$ $H \times K$
	$1 \cdot 1$	$(1, 1)$
1		
a	$a^4 a^3$	$(a^4\ a^3)$
a^2	$a^2\ 1$	$(a^2, 1)$
a^3	$1\ a^3$	$(1, a^3)$
a^4	$a^4\ 1$	$(a^4, 1)$
a^5	$a^2\ a^3$	(a^2, a^3)

The external and internal direct product are almost the same thing.

HK internal direct product, H, K regarded as subgroups of G

$H \times K$ external direct product H, K regarded as distinct entities.

Section 5.5 Semi-Direct Products

The semi direct product $H \rtimes G$ is a generalization of $H \times K$.

Definition Suppose H, K are groups, and $\varphi : K \longrightarrow Aut(H)$ is a homomorphism. For $k \in K$, $h \in H$, write $\varphi(k)(h) = k.h$. $\left\{ \begin{matrix} 1.h = 1 \\ k.1 = 1 \end{matrix} \right\}$

The semidirect product $H \rtimes_\varphi K$ is the set

$$H \rtimes_\varphi K = \{ (h, k) \mid h \in H, k \in K \}$$

with operation $(h, k)(h' k') = (h k.h', kk')$

This is a group. Check multiplication is associative

Identity: $(1,1)$ Because $(h, k)(1,1) = (h\, k.1, k1) = (h, k)$

$(1\ 1)(h,k) = (1\, 1.h, 1k) = (h, k)$

Inverse: $(h,k)^{-1} = (k^{-1}.h^{-1}, k^{-1})$ because $(h,k)(k^{-1}.h^{-1}, k^{-1})$

$= (h\, k.k^{-1}.h^{-1}, kk^{-1})$

$= (h\, (kk^{-1}).h^{-1}, 1) = (1,1)$

Example: $H = \mathbb{Z}_n = \langle r \rangle = \{1, r, r^2, \dots, r^{k-1}\}$

$K = \mathbb{Z}_2 = \langle s \rangle = \{1, s\}$

Let $\mu : H \to H$ be automorphism $\mu(x) = x^{-1}$

$\langle \mu \rangle = \{1, \mu\} \leq Aut(H)$

Homomorphism $\cancel{\varphi : K \to Aut(H)}$

$\varphi : K \longrightarrow \langle \mu \rangle \leq Aut(H)$ $\begin{cases} \varphi(1) = 1 \\ \varphi(s) = \mu \end{cases}$

$$H \rtimes_\varphi K = \{ (1,1), (r,1), (r^2,1) \dots, (r^{n-1}, 1), (1, s), (r, s), (r^2 s) \dots (r^{n-1}, s) \}$$

$$D_{2n} = \{ 1, \quad r, \quad r^2 \dots \quad r^{n-1}, \quad s, \quad rs, \quad r^2 s, \dots \quad r^{n-1} s \}$$

$H \rtimes_\varphi K$	$(r,1)^n = (1,1)$	$(r, s)^2 = (1,1)$	$(1,s)(r^k,1) = (1\, s.r^k, s) = (r^{-k}, s) = (r^{-k}, 1)(1,s)$
D_{2n}	$r^n = 1$	$s^2 = 1$	$s\ r = \dots\dots = r^{-k} s$

Conclusion: $H \rtimes_\varphi K \cong D_{2n}$