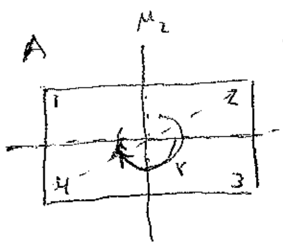
# Section 1.2 Dihedral Groups

Important class of groups: Symmetries of geometric objects

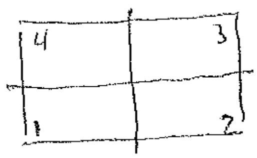
> A *symmetry* or *rigid motion* of a geometric object $A$ is a map $\mu : A \to A$ that does not deform $A$.

Example: symmetries of a rectangle $A$.

$1(A)$

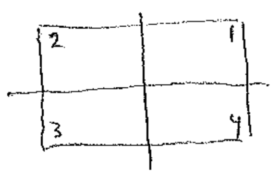



$1$: $1 \to 1$, $2 \to 2$, $3 \to 3$, $4 \to 4$



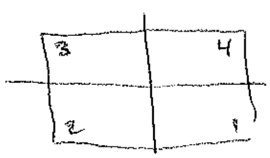

$\mu_1(A)$

$\mu_1$: $1 \to 4$, $2 \to 3$, $3 \to 2$, $4 \to 1$




$\mu_2(A)$

$\mu_2$: $1 \to 2$, $2 \to 1$, $3 \to 4$, $4 \to 3$




$r(A)$

$r$: $1 \to 3$, $2 \to 4$, $3 \to 1$, $4 \to 2$

| | $1$ | $\mu_1$ | $\mu_2$ | $r$ |
|---|---|---|---|---|
| $1$ | $1$ | $\mu_1$ | $\mu_2$ | $r$ |
| $\mu_1$ | $\mu_1$ | $1$ | $r$ | $\mu_2$ |
| $\mu_2$ | $\mu_2$ | $r$ | $1$ | $\mu_1$ |
| $r$ | $r$ | $\mu_2$ | $\mu_1$ | $1$ |

Under composition, these symmetries form a group.

Called the *group of symmetries* of $A$.

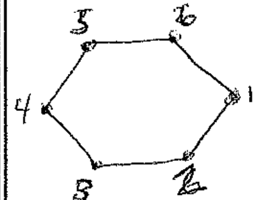
## Section 1.2

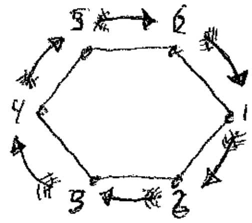
### The Dihedral Groups $D_{2n}$ — $D_2, D_4, D_6, D_8$ … etc.

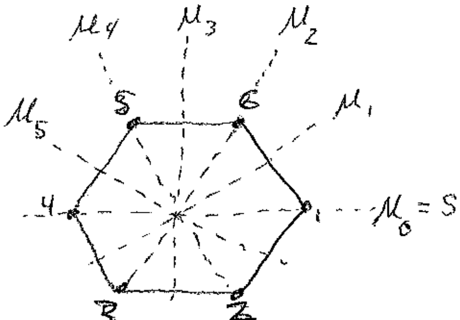
$D_{2n}$ = Group of symmetries of regular n-gon.

#### Example

$D_{12} = D_{2\cdot 6}$ = Group of symmetries of hexagon.







$$r = \begin{pmatrix} \text{rotation} \\ \text{by } \frac{2\pi}{6} \end{pmatrix} \qquad \mu_i = \begin{pmatrix} \text{reflection across} \\ \text{line at angle } \frac{\pi}{6} i \\ \text{with } s\text{-axis} \end{pmatrix}$$

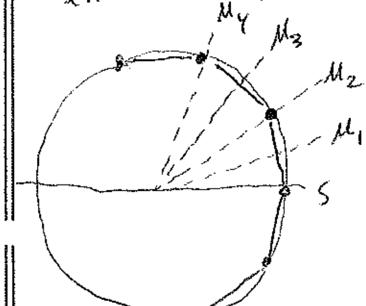
$$r^i = \underbrace{rr\cdots r}_{i} = \begin{pmatrix} \text{rotation} \\ \text{by } \frac{2\pi}{6} i \end{pmatrix} \qquad \mu_i^2 = 1$$

$$r^6 = 1 \qquad \mu_i^{-1} = \mu_i$$

$$D_{12} = \{ 1 \;\; r \;\; r^2 \;\; r^3 \;\; r^4 \;\; r^5 \;\; s \;\; \mu_1 \;\; \mu_2 \;\; \mu_3 \;\; \mu_4 \;\; \mu_5 \}$$

#### In General

$D_{2n}$ = Group of symmetries of regular n-gon.



$$r = \begin{pmatrix} \text{rotation by} \\ \frac{2\pi}{n} \end{pmatrix}$$

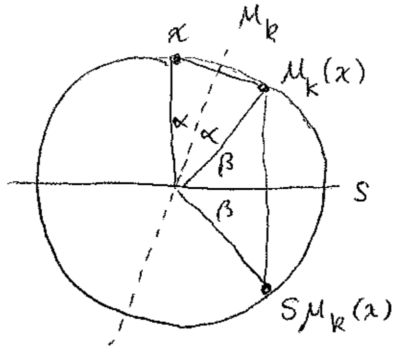
$$\mu_i = \begin{pmatrix} \text{reflection across} \\ \text{line at angle } \frac{\pi i}{n} \\ \text{with } s\text{-axis} \end{pmatrix}$$

$$D_{2n} = \{ 1, r, r^2, r^3, \ldots r^{n-1}, s, \mu_1, \mu_2 \cdots \mu_{n-1} \}$$

Note: $|D_{2n}| = 2n$

But how does multiplication work out? e.g. $\mu_j \mu_k = ?$

Key to answer: Work out $s\mu_k$




$$s\mu_k = \begin{pmatrix} \text{rotation by} \\ 2\alpha + 2\beta = \\ 2(\alpha+\beta) = \\ 2\left(\frac{\pi}{n}k\right) = \frac{2\pi}{n}k \end{pmatrix} = r^k$$

$$\boxed{s\mu_k = r^k}$$

$$s\mu_k = r^k$$
$$\mu_k = s^{-1}r^k$$
$$\boxed{\mu_k = sr^k}$$
$$\mu_k^{-1} = r^{-k}s^{-1}$$
$$\boxed{\mu_k = r^{-k}s}$$

$$\boxed{\mu_k = sr^k}$$
$$\boxed{sr^k = r^{-k}s}$$

Update:

$$D_{2n} = \{1, r, r^2, \dots r^{n-1}, s, \mu_1, \mu_2 \dots \mu_{n-1}\}$$

$$\boxed{D_{2n} = \{1, r, r^2, \dots r^{n-1}, s, sr, sr^2, \dots, sr^{n-1}\}}$$

Multiplication in $D_{2n}$:

$(sr^k)(sr^l) = r^{-k}ssr^l = r^{-k}r^l = r^{l-k} \pmod n$

$r^k(sr^l) = (r^k s)r^l = sr^{-k}r^l = sr^{l-k} \pmod n$

$r^k r^l = r^{k+l}$.

From this you can work out the entire mult. table.

## Generators and Relations

$$D_{2n} = \{1, r, r^2, \dots r^{n-1}, s, sr, sr^2, \dots sr^{n-1}\}$$

**Note** Every element of $D_{2n}$ is a product of $r$'s and $s$'s. We say $r$ and $s$ are <u>generators</u> of $D_{2n}$.

They obey the relations $r^n = 1$, $s^2 = 1$, $rs = sr^{-1}$ and the entire multiplication structure can be worked out through these relations.

We write
$$D_{2n} = \langle \underbrace{r, s}_{\text{generators}} \mid \underbrace{r^n = 1, s^2 = 1, rs = sr^{-1}}_{\text{relations}} \rangle$$

**Example** $\langle r \mid r^5 = 1 \rangle = \{1, r, r^2, r^3, r^4\} \cong \mathbb{Z}/5\mathbb{Z}$.

**Example**
$$\begin{aligned} \langle a, b \mid ab = ba \rangle &= \{a^k b^l \mid k, l \in \mathbb{Z}\} \\ &\cong \{(a^k, b^l) \mid k, l \in \mathbb{Z}\} \\ &\cong \mathbb{Z} \times \mathbb{Z}. \end{aligned}$$

In general we can always describe a group as

$$G = \langle \underset{\substack{\uparrow \\ \text{set of} \\ \text{generators}}}{S} \mid \underbrace{R_1 \ R_2 \dots R_n}_{\text{relations}} \rangle$$

**Exercise** Describe the group $G = \langle a, b \mid a^2 = 1, b^2 = 1 \rangle$.

$$G = \{1, a, b, ab, ba, aba, bab, abab, baba, \dots\}$$

$$(abab)(bab) = abab^2ab = ab\,a^2 b = ab^2 = a$$

etc

$$(abab)^{-1} = baba$$