Section 4.6 The Simplicity of A_n.

Goal: Show A_n is simple for any $n \neq 4$.

$A_2 \leq S_2 \quad A_2 = \{1\}$ (clearly simple)

$A_3 \leq S_3 \quad A_3 = \{1, (123)\,(132)\} \cong \mathbb{Z}_3$ (simple)

$A_4 \leq S_4 \quad A_4 = \{1, (12)(34), (13)(24), (14)(23)$
$(123), (124), (134), (234)$
$(132), (142), (143), (243)\}$

Note subgroup $V = \{1\ (12)(34), (12)(24), (14)(23)\}$ is normal in A_4 so $\boxed{A_4 \text{ is not simple}}$

Theorem 12: A_5 is simple

Theorem 24: If $n \geq 5$ then A_n is simple.

Proof uses induction on n, with Theorem 12 serving as the base case.

Chapter 5 Direct and Semidirect Products

Basic Themes
- Combining groups to form new groups
- Decomposing groups into products of simpler groups
- Understanding structure of complex groups by viewing them as products of simpler groups.

Section 5.1 Direct Products

Definition If $G_1 \; G_2 \cdots G_n$ are groups, their *direct product* is the group $G_1 \times G_2 \times \cdots \times G_n$ with operation

$$(g_1, g_2, \cdots, g_n)(h_1, h_2, \cdots, h_n) = (g_1h_1, g_2h_2, \cdots g_nh_n)$$

direct product operation

operation in G_2, etc

Example $G = (\mathbb{Z}/4\mathbb{Z}) \times GL_2(\mathbb{R})$

$$\left(3, \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right)\left(2, \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix}\right) = \left(3+2, \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix}\right) = \left(1, \begin{bmatrix} 4 & 5 \\ 8 & 11 \end{bmatrix}\right)$$

Identity of G is $\left(0, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$.

Example $V_4 = Z_2 \times Z_2 \qquad \mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$

Proposition 6

1. $Z_m \times Z_n = Z_{mn} \iff \gcd(m, n) = 1$
2. If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ (prime factorization, then

$$Z_n = Z_{p_1^{\alpha_1}} \times Z_{p_2^{\alpha_2}} \times \cdots \times Z_{p_k^{\alpha_k}}.$$

Normal subgroup of $G_1 \times G_2 \times \cdots \times G_n$:

$$\{(1, 1, \cdots, x, \cdots, 1, 1) \mid x \in G_i\} \cong \{1\} \times \{1\} \times \cdots \times \{1\} \times G_i \times \{1\} \times \cdots \times \{1\} \cong G_i$$

Projection homomorphisms

$$\pi_i : G_1 \times G_2 \times \cdots \times G_i \times \cdots \times G_n \longrightarrow G_i$$

$$\pi_i(g_1 \; g_2 \cdots g_i \cdots g_n) = g_i$$

$$\operatorname{Ker} \pi_i = \{(g_1, g_2, \cdots 1 \cdots g_n)\} \cong G_1 \times G_2 \times \cdots \times G_{i-1} \times G_{i+1} \times \cdots \times G_n$$

In the next section we will use these ideas to classify and understand <u>all</u> finite abelian groups.