Section 4.5 Sylow's Theorems (Continued)

Sylow's Theorem Suppose $|G| = p^\alpha m$, p prime and $p \nmid m$.

① G has a Sylow p-subgroup P, i.e. $|P| = p^\alpha$.

② Any two Sylow p-subgroups are conjugate in G;
If P is Sylow and Q is a p-subgroup then $Q \leq gPg^{-1}$, $g \in G$.

③ $n_p(G) = (\#\text{ of Sylow subgroups}) \equiv 1 \pmod p$ and
$n_p(G) = |G : N_G(P)| = \dfrac{|G|}{|N_G(P)|}$. Hence $n_p \mid m$.

Example Suppose $|G| = pq$ for p, q prime, $p < q$.
Then G has normal Sylow q-subgroup G. If $p \nmid (q-1)$,
then G is cyclic.

Proof: Let $Q \in Syl_q(G)$. Then $n_q(G) = 1 + kq$, divides p
thus $n_q(G) = 1$. Q is only Sylow-p subgroup; $Q \trianglelefteq G$.
Suppose $p \nmid (q-1)$. Let $P \in Syl_p(G)$. Then $n_p = 1 + lp$, $n_p \mid q$
so $n_p = 1$ or q; if q, $q = 1 + lp$, $q - 1 = lp$, $p \mid (q-1)$.
Hence $n_p = 1$ so $P \trianglelefteq G$.

Then $\underbrace{G / C_G(P)}_{\text{order } 1 \text{ or } q} \cong H \leq Aut(P) = Aut(\mathbb{Z}_p)$ (Prop. 13)

{order $p-1$}

Therefore $C_G(P) = G$

Define $\varphi : P \times Q \to G$ as $\varphi(x,y) = xy$

Homomorphism: $\varphi((x,y)(x'y')) = \varphi(xx', yy') = xx'yy'$
$= xy\,x'y' = \varphi(x,y)\,\varphi(x'y')$

Also $\ker \varphi = 1$ because $\varphi(x,y) = xy = 1 \Rightarrow x = y^{-1} \in P$
so $y \in P \Rightarrow y = 1$, $x = 1$.

Thus φ is isomorphism

$G \cong P \times Q \cong \mathbb{Z}_p \times \mathbb{Z}_q \cong \mathbb{Z}_{pq}$. is cyclic.

Consequence Any group of order $15 = 3 \cdot 5$ is cyclic.
Same for any group of order 35, 33, 77, etc.

	2	3	5	7	11
2					
3		·	15	21	33
5	~~10~~				
7					

	2	3	5	7	11
2					
3			15		33
5				35	
7					77

<u>Example</u> Show that any group of order 132 is not simple (i.e. that it has a normal subgroup.)

$132 = 2^2 \cdot 3 \cdot 11$

Let $P \in Syl_{11}(G)$.

Then $n_{11} = 1, 12, \overbrace{\cancel{23}, \cancel{34}, \cancel{45}, \cancel{56}, \cancel{67}, \cancel{78}}^{\text{don't divide } 2^3 \cdot 3 = 12}$

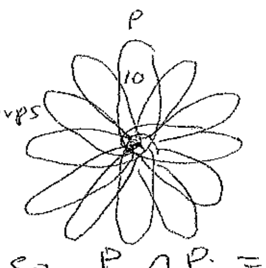
i.e. $n_{11} = 1$ or 12

If $n_{11} = 1$, then $P \trianglelefteq G$ and we are done.

Otherwise $n_{11} = 12$

Call these 12 subgroups

$P_1 \; P_2 \; P_3 \; \ldots \; P_{12}$



Each is $P_i \cong \mathbb{Z}_{11}$ so $P_i \cap P_j = 1 \quad \forall i \neq j$.

Each non-identity element in P_i has order 11.

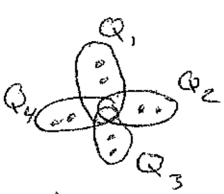
$\left| \bigcup_{i=1}^{12} P_{11} \right| = 1 + 10 \cdot 12 = 121$

This union includes 1 and 120 elements of order 11.

There are $132 - 121 = 11$ elements remaining.

Let $Q \in Syl_3(G)$.

Then $n_3 = 1 + 3k = 1, 4, 8, \ldots$



If $n_3 = 1$ then $Q \trianglelefteq G$ and we're done. Otherwise $n_3 > 1$.

Can't have $n_3 = 8$, this would give $2 \cdot 8 = 16$ elements of order 3.

Thus $n_3 = 4$, and we have 8 elements of order 3.

Now there are $11 - 8 = 3$ elements remaining.

All accounted for have orders 1, 11 or 3.

Let $R \in Syl_2(G)$. $|R| = 2^2 = 4$. Each element of R has order 1, 2 or 4. R has 3 elements of order 2 or 4.

These must be the missing elements.

Thus there is only room for 1 Sylow 2-subgroup.

Hence $R \trianglelefteq G$.

Other Examples From Text

Ex If $|G| = 12$ then either $G \cong A_4$ or $\exists\, H,\ H \trianglelefteq G,\ |H| = 3$.

Ex If $|G| = p^2 q$ then $\exists\, H \trianglelefteq G$ for either $|H| = p^2$ or $|H| = q$.

Groups of order 60

Proposition 21 If $|G| = 60 = 2^2 \cdot 3 \cdot 5$ and ~~Syl~~ $n_5(G) > 1$ then G is simple.

Corollary 22 A_5 is simple

Proof

$\langle(12345)\rangle = \{1\ (12345),\ (13524),\ (14253),\ (15432)\}$

$\langle(13245)\rangle = \{1\ (13245) \ldots\}$

are distinct sylow 5-subgroups. Apply Proposition 21

Proposition 23 If $|G| = 60$ and G is simple, then $G \cong A_5$.