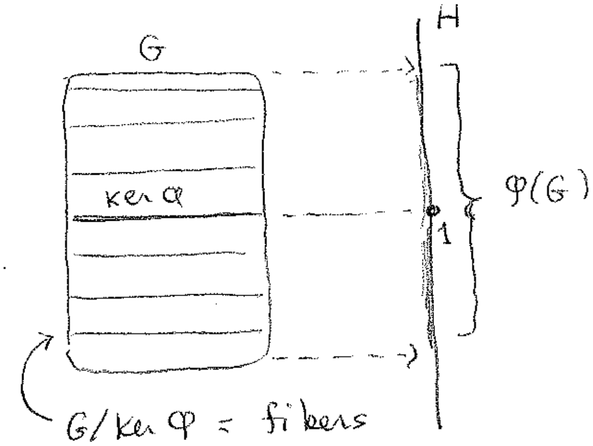
Section 3.3 Isomorphism Theorems

Theorem 16 (First Isomorphism Theorem)

If $\varphi : G \to H$ is a homomorphism, then $\ker \varphi \trianglelefteq G$ and $G/\ker\varphi \cong \varphi(G)$. Moreover:

$$\begin{tikzcd}
g \arrow[rr, mapsto, dashed] \arrow[dd, mapsto, dashed] & & \varphi(g) \\
G \arrow[r, "\varphi"] \arrow[d, "\pi"'] & \varphi(G) \leq H & \\
G/\ker\varphi \arrow[ur, "\sim"'] & & \\
g\ker\varphi \arrow[uurr, mapsto, dashed] & &
\end{tikzcd}$$



The proof is just a re-hashing of the outlook developed in Section 3.1. The more you use the theorem, the more transparant it becomes.

Corollary 17 If $\varphi: G \to H$ is a homomorphism, then

① φ injective $\iff \ker\varphi = \{1\}$

② $|G : \ker\varphi| = |\varphi(G)|$

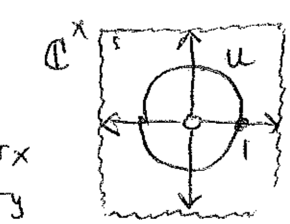
Theorem 16 is very useful for computing factor groups.

Problem Given $K \trianglelefteq G$, what is G/K?

Solution Find group U and surjective homomorphism $\varphi : G \to U$ with kernel K. Then $G/K \cong U$.

Example $\mathbb{Z} \trianglelefteq \mathbb{R}$ because $\mathbb{R}$ is abelian. What is $\mathbb{R}/\mathbb{Z}$?

Solution: Let $U = \{z \in \mathbb{C} \mid |z| = 1\} \leq \mathbb{C}^\times$ be the unit circle in $\mathbb{C}^\times$.



Put $\varphi : \mathbb{R} \to U$ $\quad \varphi(x) = e^{i2\pi x} = \cos 2\pi x + i \sin 2\pi x$

Homomorphism: $\varphi(x+y) = e^{i2\pi(x+y)} = e^{i2\pi x + i2\pi y}$
$= e^{i2\pi x} e^{i2\pi y} = \varphi(x)\varphi(y)$

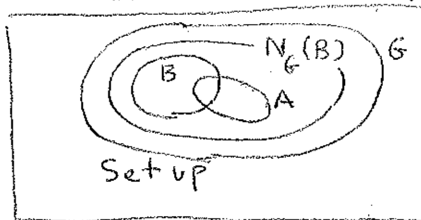
Therefore $\boxed{\mathbb{R}/\mathbb{Z} \cong U}$

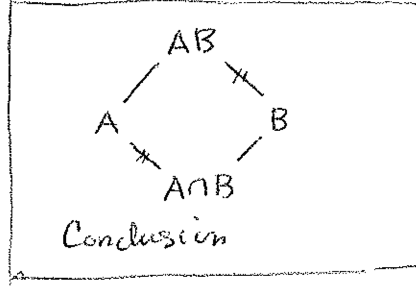
Theorem 18 (Second or Diamond Isomorphism Theorem)

Suppose $A, B \leq G$ and $A \leq N_G(B)$

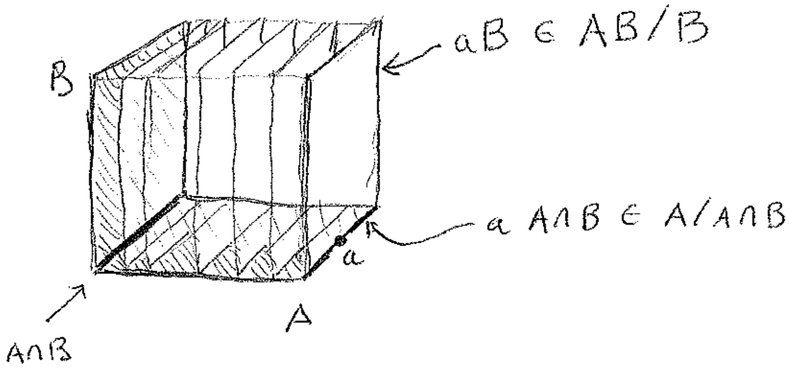
Then: $AB \leq G$, $B \trianglelefteq AB$, $\cancel{A \trianglelefteq A \cap B}$ $A \cap B \trianglelefteq A$

and $AB/B \cong \cancel{A \cap A B}$ $A/A\cap B$





Intuitive Picture:



Cosets in AB/B correspond to Cosets in $A/A\cap B$

$aB \longleftrightarrow a\,A\cap B$

Idea of proof: Define $\varphi : A \longrightarrow AB/B$
$\varphi(a) = aB$ (surjective homo)

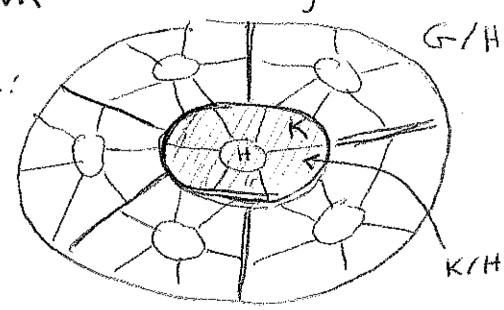
Then $\ker \varphi \cong A \cap B$ so $A/\ker\varphi = A/A\cap B \cong AB/B$. ∎

Theorem 19 (Third Isomorphism Theorem)

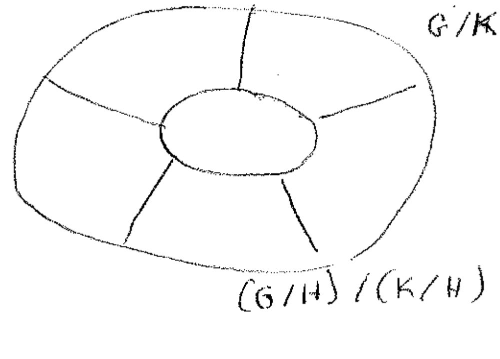
If $H, K \trianglelefteq G$, ~~thand~~ $H \leq K$, then:

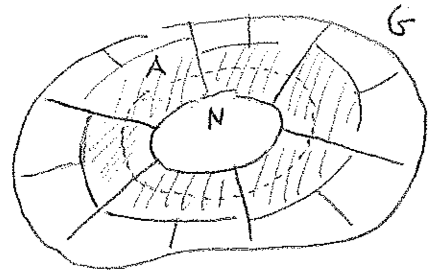
$K/H \trianglelefteq G/H$ and

$(G/H)/(K/H) \cong G/K$



Intuitive picture:





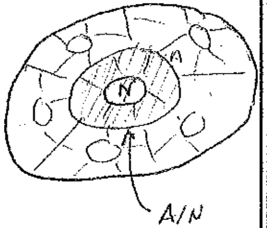
<u>Notation</u> Suppose $N \trianglelefteq G$

Given $A \le G$, with $N \le A$

then $\underset{= A/N}{\overline{A}} = \{aN \mid a \in A\} \le G/N$

<u>Proof</u> ① $1 \in A$, so $1N \in \overline{A}$.

② $aN, bN \in \overline{A} \Rightarrow a, b \in A \Rightarrow ab^{-1} \in A \Rightarrow (aN)(bN)^{-1} = ab^{-1}N \in \overline{A}$



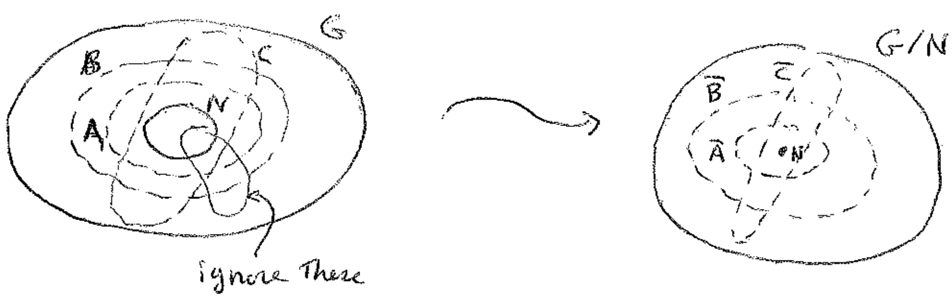
<u>Note</u> $\overline{A} = A/N \le G/N$

<u>Theorem 20</u> (Fourth Isomorphism Theorem)

Suppose $N \trianglelefteq G$. Then there is a bijection

$$\{A \le G \mid N \le A\} \longrightarrow \{\text{subgroups of } G/N\}$$

$$A \longmapsto \overline{A}$$



Moreover, for $A, B \le G$ and $N \le A$, $N \le B$

① $A \le B \iff \overline{A} \le \overline{B}$

② $A \le B \implies |B : A| = |\overline{B} : \overline{A}|$

③ $\overline{\langle A, B\rangle} = \langle \overline{A}, \overline{B} \rangle$

④ $\overline{A \cap B} = \overline{A} \cap \overline{B}$

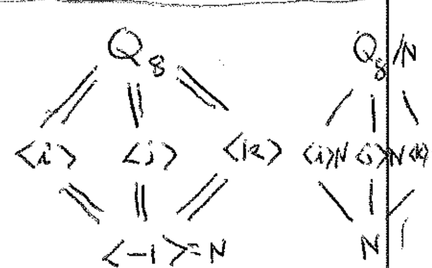
⑤ $A \trianglelefteq G \iff \overline{A} \trianglelefteq \overline{G}$

<u>Example</u> $N = \langle -1 \rangle = \{1, -1\} \trianglelefteq Q_8$

$\langle -1 \rangle = N \le \langle i \rangle = \{1, i, -1, -i\}$

$N \le \langle j \rangle$

$N \le \langle k \rangle$

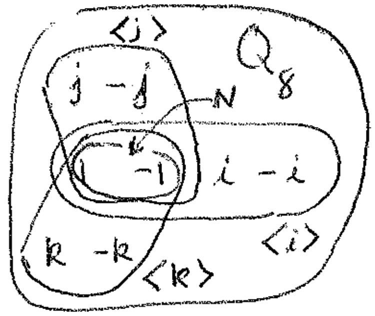


<u>Note</u> Subgroup lattice for Q_8/N is same as lattice for V_4. In fact $Q_8/N \cong V_4$

<u>Theorem 20</u> $\begin{pmatrix} \text{Subgroups} \\ \text{of } G/N \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{Subgroups of} \\ G \text{ containing } N \end{pmatrix}$

Illustration : Q_8

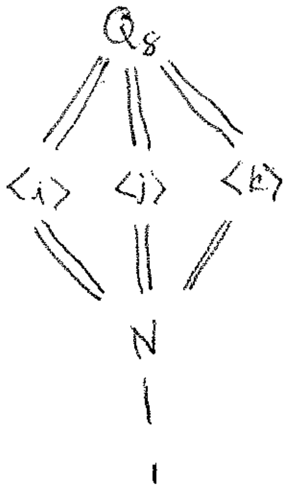
$N = \{1, -1\} = \langle -1 \rangle$

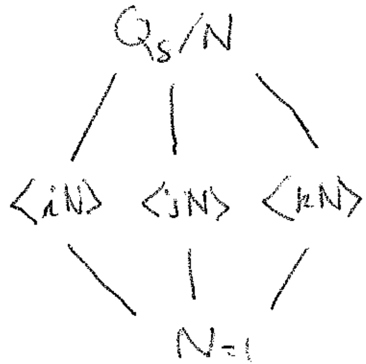


$Q_8 / N = \{ N, iN, jN, kN \}$

	N	iN	jN	kN
N	N	iN	jN	kN
iN	iN	N	kN	jN
jN	jN	kN	N	iN
kN	kN	jN	iN	N

$Q_8 / N = V_4 = \begin{pmatrix} \text{Klein} \\ \text{4-group} \end{pmatrix}$





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