MATH 601
Abstract Algebra I

Richard Hammack

www.people.vcu.edu/∼rhammack/Math601/

Today: Chapter 0, Section 1.1

Goal: Establish notation; recall elemental ideas and the definition of a group; introduce groups of symmetries.

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  Example: $a = -11$, $b = 4$; $-11 = -3 \cdot 4 + 1$

Math 601 Mantra

$$\boxed{\text{NEVER UNDERESTIMATE THE DIVISION ALGORITHM}}$$

Consequence of Division Algorithm

Theorem $\gcd(a, b) = 1 \iff \exists\, x, y \in \mathbb{Z}$ with $ax + by = 1$.

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Reversing roles of a and b, we get $k|b$.
Thus k is a common positive divisor of both a and b.

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Thus $1 \le k \le \gcd(a,b) = 1$, so $k = 1$.

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Thus k is a common positive divisor of both a and b.
Thus $1 \leq k \leq \gcd(a, b) = 1$, so $k = 1$.

($\Longleftarrow$) (Contrapositive)
Suppose $\gcd(a, b) = k > 1$.

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Then $a = kc$ and $b = kc'$ for some $c, c' \in \mathbb{Z}$.

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Thus k is a common positive divisor of both a and b.
Thus $1 \leq k \leq \gcd(a, b) = 1$, so $k = 1$.

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Suppose $\gcd(a, b) = k > 1$.
Then $a = kc$ and $b = kc'$ for some $c, c' \in \mathbb{Z}$.
Thus for any integers x, y, we have
$ax + by = kcx + kc'y = k(cx + c'y) \neq 1$. ■

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$a \equiv b \pmod{n}$ provided $n|(a-b)$.

Example: $\mathbb{Z}/3\mathbb{Z}$

Equivalence classes:
$\overline{0} = \{x \in \mathbb{Z} \mid 3|(x-0)\} = \{3k+0 \mid k \in \mathbb{Z}\} = \{\ldots, -3, 0, 3, 6, \ldots\}$
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Addition on equivalence classes: $\overline{a} + \overline{b} = \overline{a+b}$

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Operations associative: $(\overline{a}\,\overline{b})\overline{c} = (\overline{ab})\overline{c} = \overline{(ab)c} = \overline{a(bc)} = \overline{a}(\overline{bc}) = \overline{a}(\overline{b}\,\overline{c})$

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Definition: A *group* is a set G with a binary operation $\star : G \times G \to G$.
Abbreviation: $\star(a, b) = a \star b$.
This is required to satisfy the following three axioms:

(i) $\star$ is associative: $\quad (a \star b) \star c = a \star (b \star c) \qquad \forall a, b, c \in G$.

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Add.	$a + b$	0	$-a$	$na = \underbrace{a + a + \cdots + a}_{n}$ $0a = 0$	$ma + na = (m+n)a$ $m(na) = (mn)a$ $(-m)a = m(-a)$

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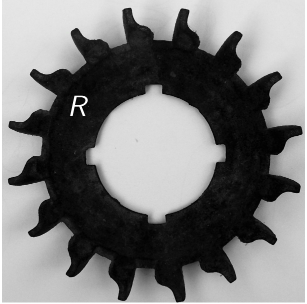

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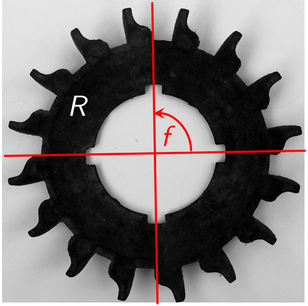

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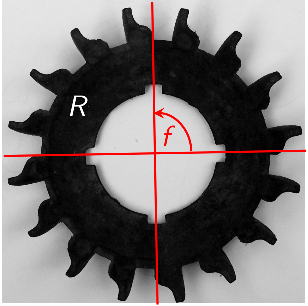

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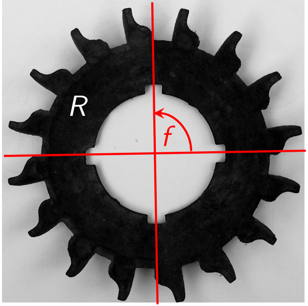

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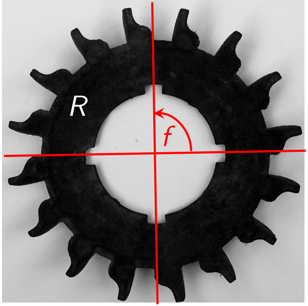

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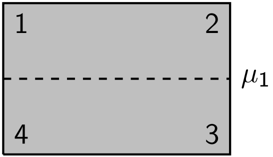
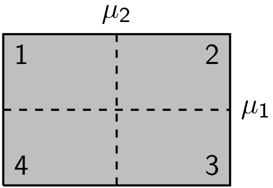
In above example $G = \{1, f, f^2, f^3\} = \{f^0, f^1, f^2, f^3\} \cong \mathbb{Z}/4\mathbb{Z}$.

Symmetry group of a rectangle

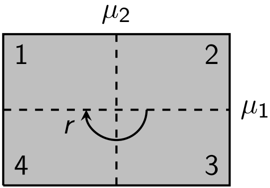
Symmetry group of a rectangle



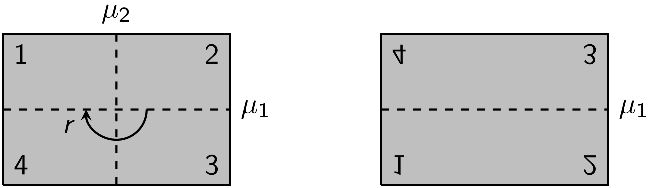
Symmetry group of a rectangle



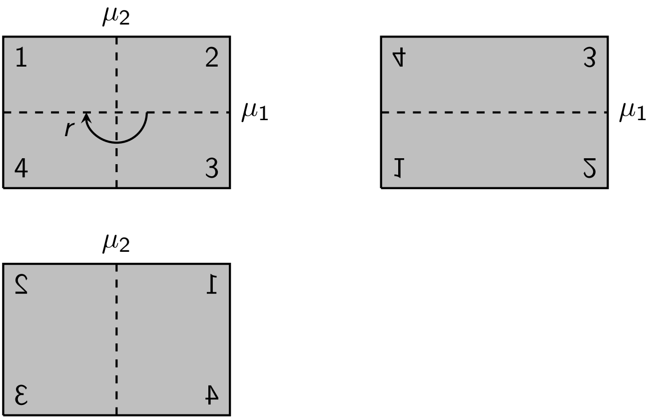
Symmetry group of a rectangle



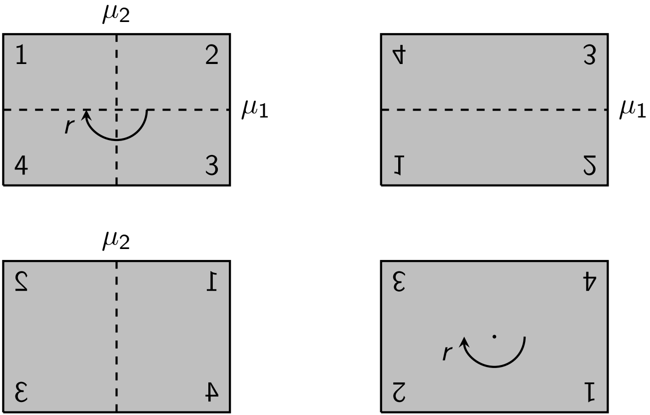
Symmetry group of a rectangle



Symmetry group of a rectangle

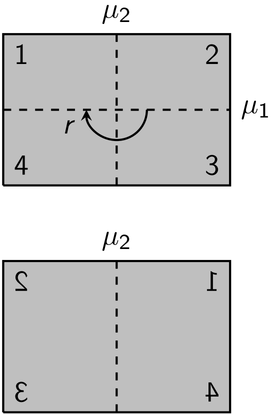


Symmetry group of a rectangle



Symmetry group of a rectangle

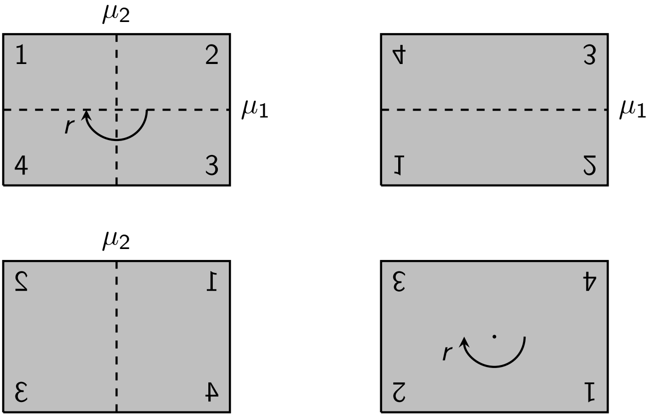
$$\mu_1^2 = 1$$



Symmetry group of a rectangle

$$\mu_1^2 = 1$$

$$\mu_2^2 = 1$$

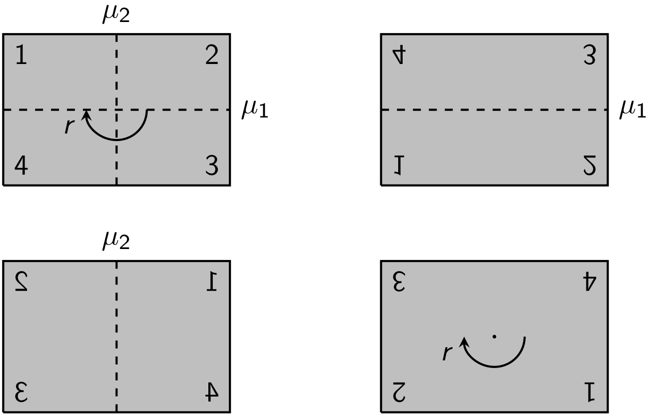


Symmetry group of a rectangle

$\mu_1^2 = 1$

$\mu_2^2 = 1$

$r^2 = 1$



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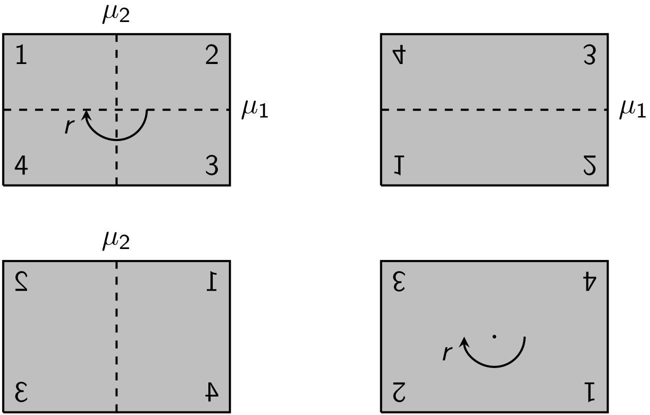


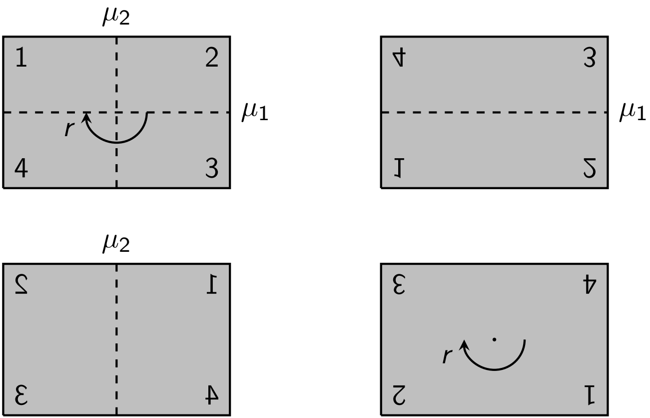
$\circ$	1	μ_1	μ_2	r
1				
μ_1				
μ_2				
r				

Symmetry group of a rectangle

$$\mu_1^2 = 1$$

$$\mu_2^2 = 1$$

$$r^2 = 1$$



$\circ$	1	μ_1	μ_2	r
1	1	μ_1	μ_2	r
μ_1	μ_1			
μ_2	μ_2			
r	r			

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$$\mu_1^2 = 1$$

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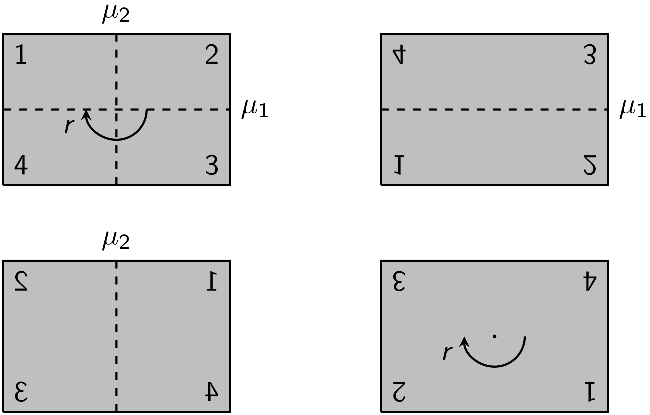
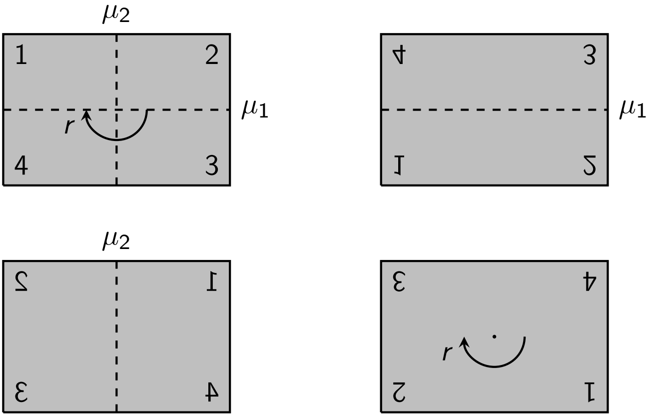


$\circ$	1	μ_1	μ_2	r
1	1	μ_1	μ_2	r
μ_1	μ_1	1		
μ_2	μ_2		1	
r	r			1

Symmetry group of a rectangle



$$\mu_1^2 = 1$$

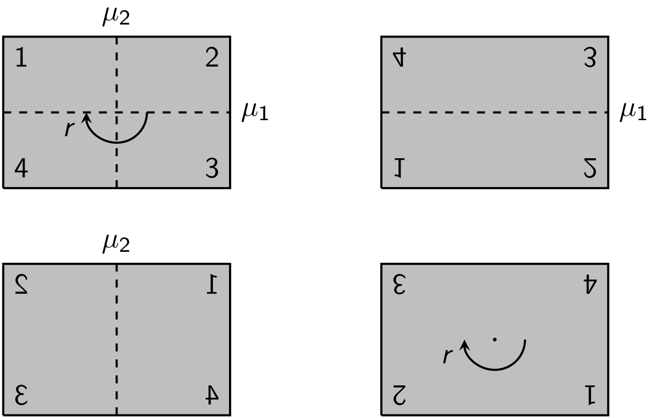
$$\mu_2^2 = 1$$

$$r^2 = 1$$

$$\mu_2\mu_1 = r$$

$\circ$	1	μ_1	μ_2	r
1	1	μ_1	μ_2	r
μ_1	μ_1	1		
μ_2	μ_2		1	
r	r			1

Symmetry group of a rectangle



$$\mu_1^2 = 1$$

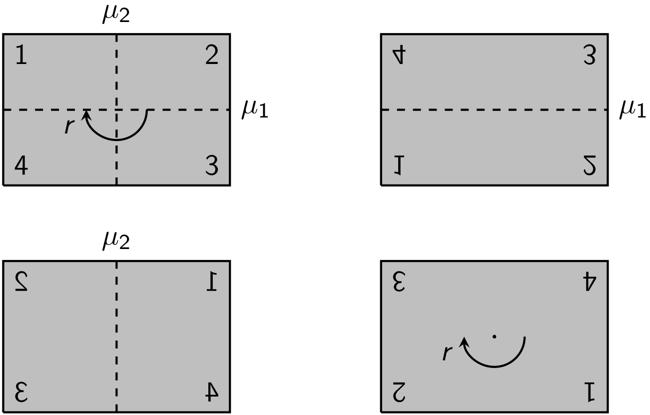
$$\mu_2^2 = 1$$

$$r^2 = 1$$

$$\mu_2 \mu_1 = r$$

$\circ$	1	μ_1	μ_2	r
1	1	μ_1	μ_2	r
μ_1	μ_1	1	r	
μ_2	μ_2	r	1	
r	r			1

Symmetry group of a rectangle



$$\mu_1^2 = 1$$

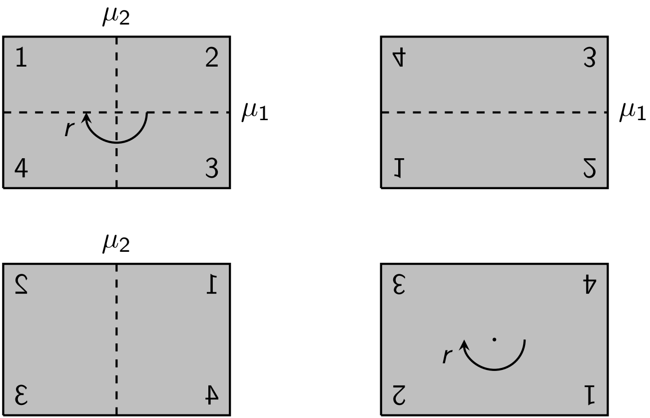
$$\mu_2^2 = 1$$

$$r^2 = 1$$

$$\mu_2 \mu_1 = r$$

$\circ$	1	μ_1	μ_2	r
1	1	μ_1	μ_2	r
μ_1	μ_1	1	r	μ_2
μ_2	μ_2	r	1	
r	r	μ_2		1

Symmetry group of a rectangle



$$\mu_1^2 = 1$$

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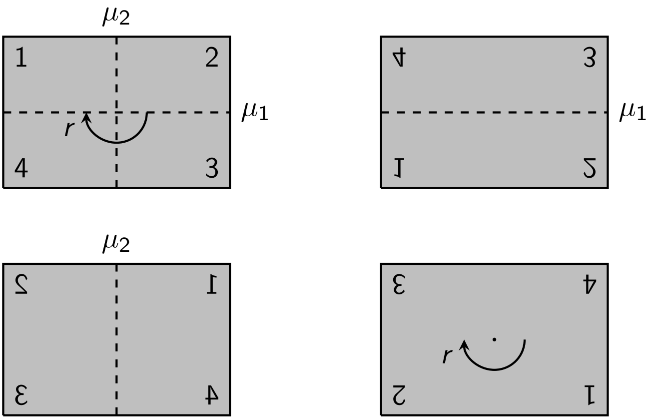
$$r^2 = 1$$

$$\mu_2\mu_1 = r$$

$\circ$	1	μ_1	μ_2	r
1	1	μ_1	μ_2	r
μ_1	μ_1	1	r	μ_2
μ_2	μ_2	r	1	μ_1
r	r	μ_2	μ_1	1

Symmetry group of a rectangle



$\mu_1^2 = 1$

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$r^2 = 1$

$\mu_2\mu_1 = r$

$\circ$	1	μ_1	μ_2	r
1	1	μ_1	μ_2	r
μ_1	μ_1	1	r	μ_2
μ_2	μ_2	r	1	μ_1
r	r	μ_2	μ_1	1

Symmetry group of rectangle is Klein 4-group $G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

Symmetry group of a frieze pattern

Symmetry group of a frieze pattern

$t_n =$ move n units horizontally

Symmetry group of a frieze pattern

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Group of symmetries:
$G = \{\ldots, t_{-3},\ t_{-2},\ t_{-1},\ t_0,\ t_1,\ t_2,\ t_3, \ldots\}$

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Multiplication: $t_m t_n = t_{m+n}$

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Multiplication: $m \star n = m + n$

$G \cong \mathbb{Z}$

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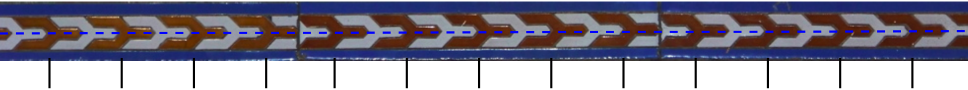
Symmetry group of a frieze pattern

$t_n =$ move n units horizontally $\qquad t_m t_n = t_{m+n}$

$h =$ horizontal reflection $\qquad h^2 = 1$

Symmetry group of a frieze pattern



$t_n =$ move n units horizontally $\qquad t_m t_n = t_{m+n}$

$h =$ horizontal reflection $\qquad h^2 = 1$ and $t_n h = h t_n$

Symmetry group of a frieze pattern

t_n = move n units horizontally $\qquad t_m t_n = t_{m+n}$

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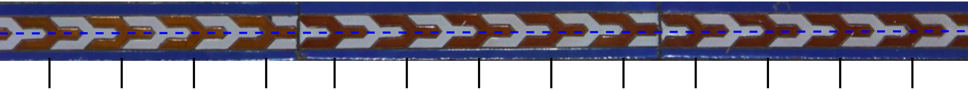
Group of symmetries:

$$G = \{\ldots, t_{-1}h^0,\ t_0h^0,\ t_1h^0,\ t_2h^0, \ldots\ \ \ldots, t_{-1}h^1,\ t_0h^1,\ t_1h^1,\ t_2h^1, \ldots\}$$

Symmetry group of a frieze pattern



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$$\{\ldots, (-1,0),\ (0,0),\ (1,0),\ (2,0), \ldots\ \ \ldots,\ (-1,1),\ (0,1),\ (1,1),\ (2,1),\ \ldots\}$$

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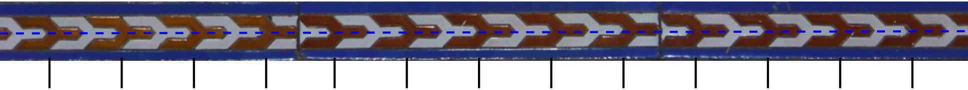

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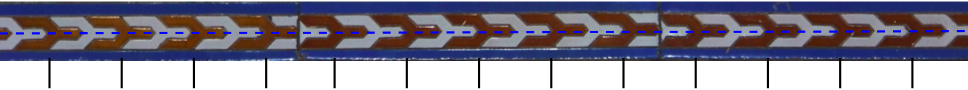
Multiplication: $(t_m h^k)(t_n h^\ell) = t_m h^k t_n h^\ell = t_m t_n h^k h^\ell = t_{m+n} h^{k+\ell (\mathrm{mod}\ 2)}$

Symmetry group of a frieze pattern



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$$(m,k)+(n,\ell) = \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (m+n,\ k+\ell (\text{mod } 2))$$

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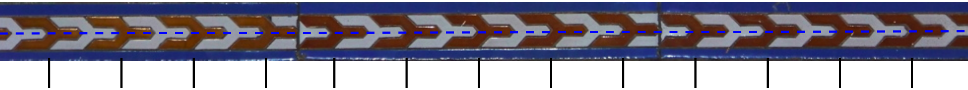

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$$(m,k)+(n,\ell) = \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (m+n,\ k+\ell (\text{mod } 2))$$

$G \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

Further Examples

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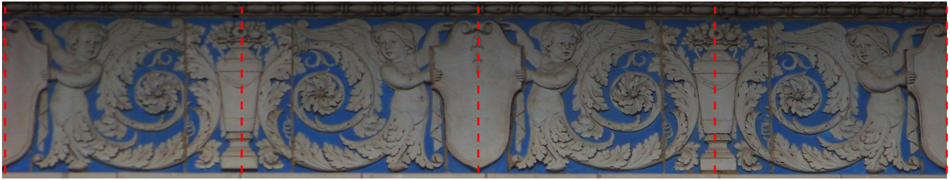
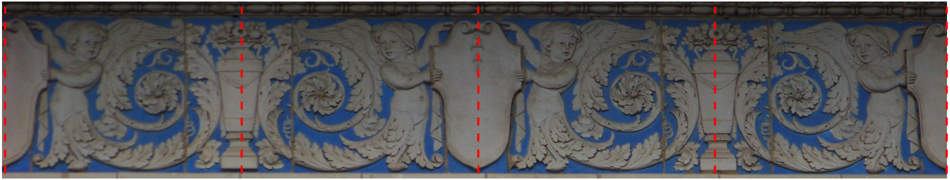
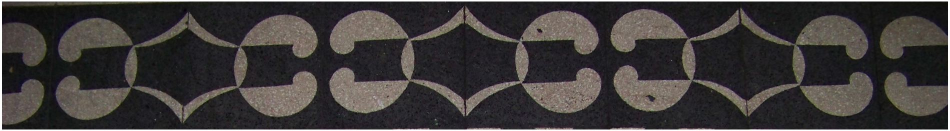
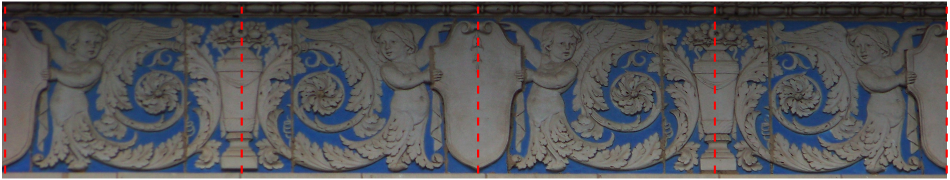


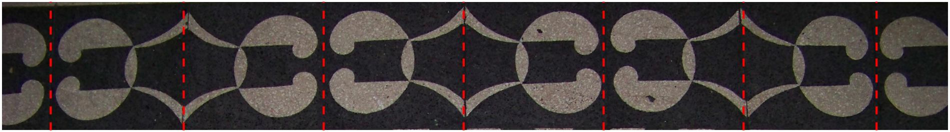
Further Examples





Further Examples





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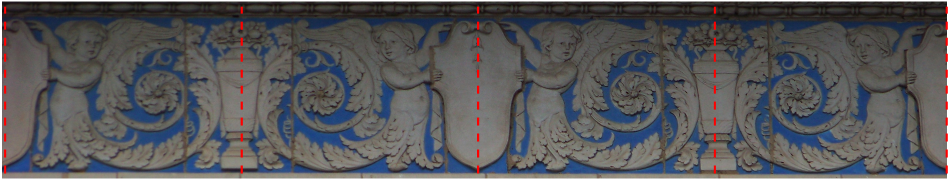

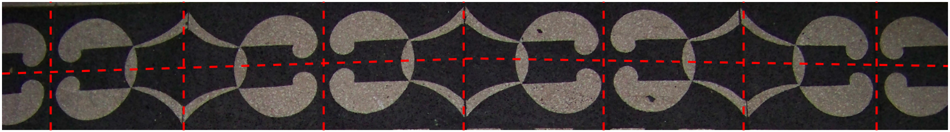

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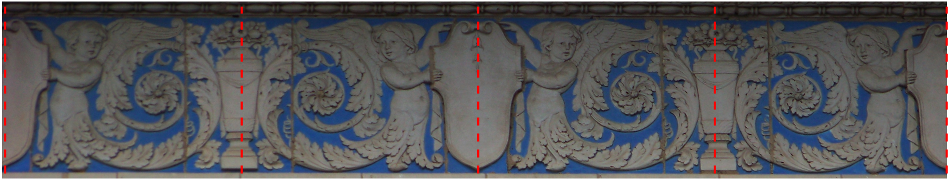

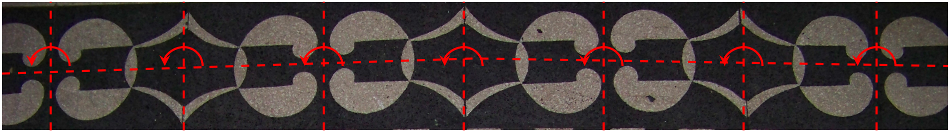

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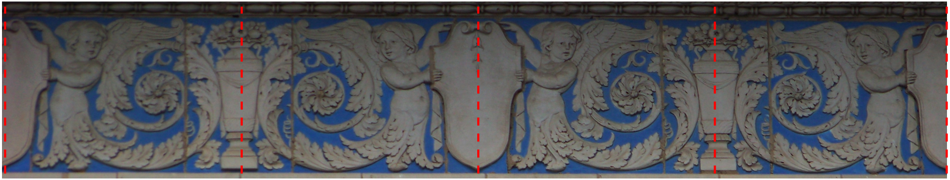

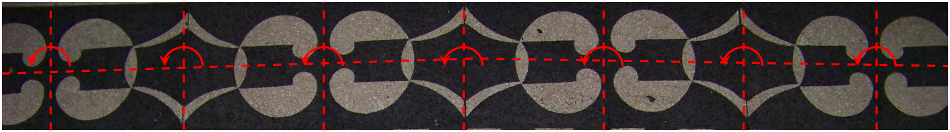

Describing groups of more complicated frieze patterns involves more sophisticated ideas in group theory (semi-direct products, etc.). This course will develop that theory, and more. Next time: **Dihedral groups**.