MATH 591: Graph Theory | Midterm | October 25, 2011

Name: ____________________ Score: ________

Directions: This is a closed-book, closed-notes test. You *may not* use calculators, computers, etc.

1. (12 points) Is the sequence $6, 6, 5, 4, 3, 2, 2$ graphical?

6	6	5	4	3	2	2
	5	4	3	2	1	1
		3	2	1	0	0
			1	0	-1	0
				N	**O**	!

2. (12 points) Prove that if G is a graph with minimum degree $\delta(G) \geq 2$ then G contains a cycle.

   Proof: Let P be a longest path in G, and let its terminal vertex be z, and let its last edge be yz. Now, as z has degree 2 or greater, there is a an edge zx with $x \neq y$. But P is a longest path by assumption, so we cannot append the edge xy to it; hence y must be a vertex that is already on P. Now traversing from y along P to z, then to x and back to y produces a cycle in G.

3. (12 points) Show that for every two vertices u, v in a connected graph G, there is a u-v walk containing all vertices of G.

   Proof: Take any two vertices u and v of G. As G is connected, there is a u-v walk W in G. If this walk happens to contain all vertices of G, we are done. Otherwise there is some vertex x not on the walk. Append to the end of the walk W a v-x walk followed by an x-v walk (which can be done since G is connected). We have now produced an extension of W that contains x. If this new walk contains all vertices of G then we are done; otherwise we can repeat the above procedure until all vertices of G are contained in a u-v walk.

4. (12 points) Construct a tree with Prüfer code (1,8,1,5,2,5).

   Solution: This is the example in the section on Prüfer codes in the text. See the text for a solution.

5. (12 points) Let G be a connected graph of order 3 or more. Prove that if $e = uv$ is a bridge of G then at least one of u or v is a cut vertex of G.

   Proof: Suppose $e = uv$ is a bridge. As G has at least one additional vertex beyond u and v, at least one of u and v is adjacent to a vertex x of G. Without loss of generality, say $ux \in E(G)$. Then removing u from G removes the edge uv, but leaves the vertices x and v, which are necessarily in different components because they were separated by the bridge. Thus u is a cut vertex.

6. (12 points) Find necessary and sufficient conditions on G and H that guarantee that the Cartesian product $G \Box H$ is Eulerian. Justify your answer.

   We show that $G \Box H$ is Eulerian if and only if all vertices of G and H have even degree or all vertices of G and H have odd degree.

   First observe that any vertex (x, y) of $G \Box H$ has degree $\deg_{G \Box H}(x, y) = \deg_G(x) + \deg_H(y)$. From this, it follows immediately that if every vertex of G and H has even degree, or if every vertex of G and H has odd degree, then every vertex of $G \Box H$ has even degree; and hence $G \Box H$ is Eulerian.

   Conversely, suppose $G \Box H$ is Eulerian, so each of its vertices (x, y) has even degree. That is, $\deg_{G \Box H}(x, y)$ is even for all $x \in V(G)$ and $y \in V(H)$. Fix a vertex $h \in V(H)$, so $\deg_{G \Box H}(x, h) = \deg_G(x) + \deg_H(h)$. Then $\deg_G(x) = \deg_{G \Box H}(x, h) - \deg_H(h)$ for every $x \in V(G)$. We infer from this that the degree of every vertex x of G has the same parity as $\deg_H(h)$. Now fix $g \in V(G)$ and as above we get $\deg_H(y) = \deg_{G \Box H}(g, y) - \deg_G(g)$ for all $y \in V(H)$. From this it follows that the degree of every vertex $y \in V(H)$ has a parity that agrees with that of $\deg_G(g)$. Conclusion: All vertices of G and H have even degree, or all have odd degree.

7. (14 points) Let G be a graph of order $n \geq 3$ having the property that for each vertex $x \in V(G)$, there is a Hamiltonian path with initial vertex x. Show that G is 2-connected but not necessarily Hamiltonian.

Proof: Suppose G is a graph of order $n \geq 3$ having the property that for each vertex $x \in V(G)$, there is a Hamiltonian path with initial vertex x. To show that G is 2-connected, we just need to show that for any $x \in V(G)$, the graph $G - x$ remains connected. Given an $x \in V(G)$, let P be a Hamilton path in G with initial vertex x. Then $G - x$ contains the path $P - x$ which meets every vertex of $G - x$. Thus $G - x$ is connected. Hence G is 2-connected.

To see that G is not necessarily Hamiltonian, consider the Petersen graph. For each of its vertices x there is a Hamiltonian path beginning at x, yet the Petersen graph is not Hamiltonian.

8. (14 points) Prove or disprove: If every vertex of a tournament belongs to a cycle, then the tournament is strong.

This is FALSE. Consider the following counterexample with two strong components (bold).
Every vertex belongs to a cycle (a 3-cycle).