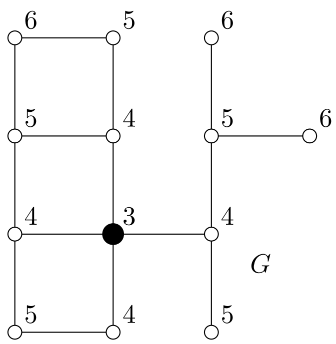
MATH 591: Graph Theory | Test | October 15, 2010

Name: ______________________ Score: ________

Directions: This is a closed-book, closed notes test. Please answer in the space provided. You *may not* use calculators, computers, etc.

1. (15 points) A graph G is drawn below. Label each vertex with its eccentricity. State the radius and diameter of G. Indicate the center of G.



Radius is 3.
Diameter is 6.
The center is the single shaded vertex with minimum eccentricity 3.

2. (15 points) Suppose $k \geq 2$. Prove that a k-regular bipartite graph has no cut-edge.

Proof. Suppose for the sake of contradiction that G is a k-regular bipartite graph ($k \geq 2$) with a cut edge ab. When ab is removed from G, the component of G containing the edge ab splits into two new components; call them A and B, with $a \in A$ and $b \in B$. Both of these components are nontrivial, since their vertices have degrees at least $k - 1 \geq 1$. Now, the component A is bipartite (since it is a subgraph of a bipartite graph), so there is a bipartition $V(A) = X \cup Y$ of A with each edge of A running between X and Y. Without loss of generality, say $a \in Y$. Then every vertex of X has degree k. By contrast every vertex of Y has degree k, *except* for the vertex a, which has degree $k - 1$. Therefore, we can count the number of edges in A in two ways:

$$\begin{aligned} |E(A)| = k|X| &= k(|Y| - 1) + (k - 1) \\ k|X| &= k|Y| - 1 \\ 1 &= k(|Y| - |X|) \\ \frac{1}{k} &= |Y| - |X| \in \mathbb{Z}. \end{aligned}$$

From the above, it follows that $k = 1$, contradicting the fact that $k \geq 2$. QED

3. (15 points) Let $k \geq 2$ be a fixed integer. Suppose a tree T has p vertices of degree k, and all the other vertices of T have degree 1. Find $n(T)$.

Proof. Sice T has p vertices of degree k and $n(T) - p$ vertices of degree 1, we have

$$\begin{aligned} 2|E(T)| &= \sum_{x \in V(T)} d(x) \\ &= p \cdot k + (n(T) - p) \cdot 1 \\ &= p(k-1) + n(T). \end{aligned}$$

But T is a tree, so $|E(T)| = n(T) - 1$, and the above calculation yields

$$\begin{aligned} 2(n(T) - 1) &= p(k-1) + n(T) \\ 2n(T) - 2 &= p(k-1) + n(T) \\ n(T) &= p(k-1) + 2 \end{aligned}$$

Therefore $\boxed{n(T) = p(k-1) + 2.}$

4. (15 points) State the following theorems carefully and precisely.

(a) Berge's Theorem

A matching M in a graph G is a maximum matching if and only if G has no M-augmenting path)

(An M-augmenting path is a path which alternates between edges in M and not in M, and whose endpoints are not saturated by M.)

(b) Hall's Theorem:

Suppose G is a bipartite graph with bipartition $V(G) = X \cup Y$.
Then G has a matching that saturates X if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$.

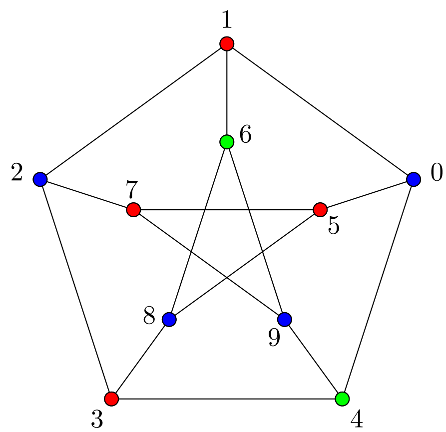
(Here $N(S)$ denotes the set of all vertices of G which are adjacent to a vertex of S.)

(c) The König-Egervary Theorem

For any bipartite graph G, the maximum size of a matching equals the minimum size of a vertex cover.

(A vertex cover is a set $Q \subseteq V(G)$ such that every edge of G has an endpoint in Q.)

5. (20 points) Find the listed invariants for the Petersen graph.



(a) $\alpha = 4$ ($\{2, 0, 8, 9\}$ is a maximum independent set.)

(b) $\gamma = 3$ ($\{3, 5, 6\}$ is a minimum dominating set.)

(c) $\alpha' = 5$ ($\{05, 16, 27, 38, 49\}$ is a perfect matching.)

(d) $\chi = 3$ ($\chi > 2$ since P has 5-cycle; See 3-coloring on left.)

(e) $\omega = 2$ (Petersen graph has K_2's but no K_3's.)

6. (10 points) Prove that $\gamma \leq \alpha$ for any graph.

Proof. Let I be a largest independent set in G, so $|I| = \alpha$. Now, if x is any vertex of G, then either $x \in I$, or x is adjacent to a vertex in I. (If x were not adjacent to a vertex in I, then we could enlarge the independent set I by appending x to it, but I is already a largest independent set. Since every vertex of G is in either in I or adjacent to a vertex in I, it follows that I is a dominating set. Since γ is the size of a smallest dominating set, we have $\gamma \leq I = \alpha$. QED

7. (10 points) Prove that $\chi \cdot \alpha \geq n$ for any graph.

Proof. Consider a coloring of G with χ colors $\{1, 2, \ldots, \chi\}$. For any color i, the set X_i of vertices with that color is an independent set in G, and therefore $|X_i| \leq \alpha$. Therefore we have

$$\begin{aligned} n &= |X_1| + |X_2| + \cdots + |X_\chi| \\ &\leq \alpha + \alpha + \cdots + \alpha \\ &= \chi\alpha. \end{aligned}$$

This establishes $\chi \cdot \alpha \geq n$. QED