VCU

MATH 525
Combinatorics

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Test 2

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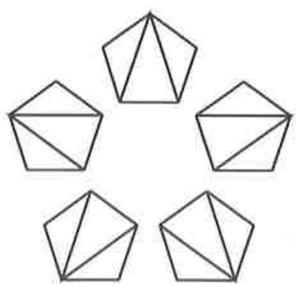
Directions. Answer the questions in the space provided. Justify each step to the extent reasonable.

This is a closed-book, closed-notes test.

There are 5 numbered questions; each is worth 20 points.



1. Solve the recurrence relation $h_n = 4h_{n-2}$ with initial values $h_0 = 0$ and $h_1 = 1$.

$$h_n - 4h_{n-2} = 0$$

$$x^n - 4x^{n-2} = 0$$

$$x^2 - 4 = 0 \leftarrow \text{(characteristic equation)}$$

$$(x-2)(x+2) = 0 \leftarrow \text{(roots } 2, -2\text{)}$$

$$h_n = a2^n + b(-2)^n$$

$$h_0 = a2^0 + b(-2)^0 = 0$$

$$h_1 = a2^1 + b(-2)^1 = 1$$

$$\Rightarrow \begin{cases} a + b = 0 \\ 2a - 2b = 1 \end{cases}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & -2 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & -1 & \frac{1}{2} \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -2 & \frac{1}{2} \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & -\frac{1}{4} \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{4} \\ 0 & 1 & -\frac{1}{4} \end{array}\right] \quad \begin{array}{l} a = \frac{1}{4} \\ b = -\frac{1}{4} \end{array}$$

Solution $h_n = \frac{1}{4} 2^n - \frac{1}{4}(-2)^n$

$$\boxed{h_n = 2^{n-2} - (-2)^{n-2}}$$

2. Solve the recurrence relation $h_n = 2h_{n-1} + n$ with initial value $h_0 = 1$.

Homogeneous part $H(n) = c\,2^n$

Particular solution $\varphi(n) = an + b$

Must satisfy:

$$\varphi(n) = 2\varphi(n-1) + n$$

$$an + b = 2(a(n-1) + b) + n$$

$$an + b = (2a+1)n + 2b - 2a$$

$a = 2a + 1 \implies \boxed{a = -1}$

$b = 2b - 2a \implies \boxed{b = 2a = -2}$

Therefore $\varphi(n) = -n - 2$

Consequently

$$h_n = H(n) + \varphi(n) = c\,2^n - n - 2$$

Then $h_0 = c\,2^0 - 0 - 2 = 1$

$C = 3$

<u>Answer</u> $\boxed{h_n = 3 \cdot 2^n - n - 2}$

3. Use generating functions to find how many ways there are to put n identical balls into four boxes, in such a way that the first box has no more than 3 balls, the second has a multiple of 4 balls, the third has at least 5 balls, and there is no restriction on the number of balls in the fourth box.

$$g(x) = \underbrace{(1+x+x^2+x^3)}_{\text{box 1}}\underbrace{(1+x^4+x^8+\cdots)}_{\text{box 2}}\underbrace{(x^5+x^6+x^7+\cdots)}_{\text{box 3}}\underbrace{(1+x+x^2+\cdots)}_{\text{box 4}}$$

$$= \frac{1-x^4}{1-x}\,\frac{1}{1-x^4}\,x^5\frac{1}{1-x}\,\frac{1}{1-x}$$

$$= \frac{x^5}{(1-x)^3} = x^5\frac{1}{(1-x)^3}$$

$$= x^5\sum_{n=0}^{\infty}\binom{n+2}{n}x^n$$

$$= \sum_{n=5}^{\infty}\binom{n-3}{n-5}x^n$$

$$= \sum_{n=0}^{\infty}\binom{n-3}{n-5}x^n$$

because for $n = 0, 1, 2, 3, 4$ we have $\binom{n-3}{n-5} = 0$

Answer: For n balls, there are $\binom{n-3}{n-5}$ ways to do this.

4. Let h_n be the number of ways to color the squares of a $1 \times n$ chessboard red, white, blue & green so there are an even number of red squares and an odd number of white ones. Find the exponential generating function for the sequence $h_0, h_1, h_2, \ldots$. Use it to find a simple formula for h_n.

$$g^{(e)}(x) = \underbrace{\left(1+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)}_{\text{red}}\underbrace{\left(x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right)}_{\text{white}}\underbrace{\left(1+x+\frac{x^2}{2!}+\cdots\right)}_{\text{blue}}\underbrace{\left(1+x+\frac{x^2}{2!}\cdots\right)}_{\text{green}}$$

$$= \frac{1}{2}(e^x+e^{-x})\frac{1}{2}(e^x-e^{-x})\,e^x e^x$$

$$= \frac{1}{4}(e^{2x}-e^{-2x})\,e^{2x}$$

$$= \frac{1}{4}(e^{4x}-1)$$

$$= \frac{1}{4}\sum_{n=1}^{\infty} 4^n \frac{x^n}{n!} - \frac{1}{4}$$

From this when $n > 0$

we have $h_n = \frac{1}{4}4^n$

i.e. $\boxed{h_n = 4^{n-1}}$

Also we have the special case $h_0 = \frac{1}{4} - \frac{1}{4} = 0$

$$\text{Thus } h_n = \begin{cases} 4^{n-1} & \text{if } n > 0 \\ 0 & \text{if } n = 0 \end{cases}$$

5. Find the (ordinary) generating function for the infinite sequence $h_0, h_1, h_2, \ldots$ defined by $h_n = \binom{n}{2}$.

Note: $\binom{n}{2} = \dfrac{n(n-1)}{2}$

Start with

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} x^n$$

Differentiate twice

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1}$$

$$\frac{2}{(1-x)^3} = \sum_{n=0}^{\infty} n(n-1) x^{n-2}$$

Now multiply both sides by $\dfrac{x^2}{2}$

$$\frac{x^2}{(1-x)^3} = \sum_{n=0}^{\infty} \frac{n(n-1)}{2} x^n$$

$$\frac{x^2}{(1-x)^3} = \sum_{n=0}^{\infty} \binom{n}{2} x^n$$

Therefore the generating function for $h_n = \binom{n}{2}$ is $\boxed{\dfrac{x^2}{(1-x)^3}}$