VCU

MATH 525
Combinatorics

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Test 1

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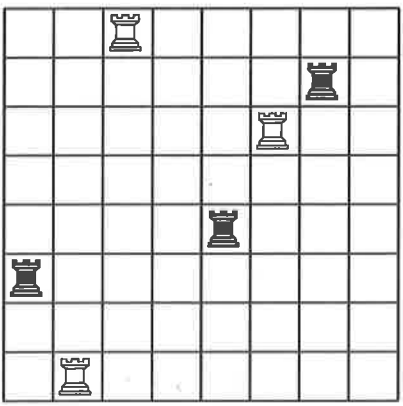
Directions. Answer the questions in the space provided. Justify each step to the extent reasonable.

This is a closed-book, closed-notes test.

There are 10 numbered questions; each is worth 10 points.



1. Consider 10-letter strings made from the letters $\overbrace{A, B, C, D, E, F, G, H}^{8}$.

(a) How many strings are there (repetition allowed) for which no two consecutive letters are the same?

$$\underline{8}\ \underbrace{\underline{7}\ \underline{7}\ \underline{7}\ \underline{7}\ \underline{7}\ \underline{7}\ \underline{7}\ \underline{7}\ \underline{7}}$$

any of the 8 letters

Any letter except the one to its left

By multiplication principle, the answer is $\boxed{8 \cdot 7^9} = 322828856$

(b) How many strings are there (repetition allowed) for which the first entry is a vowel, and the string contains exactly three E's?

These fall into two types. Let X be the set of those strings beginning with A. Let Y be those strings beginning with E.

X: $\underline{A}\ \underline{7}\ \underline{7}\ \underline{E}\ \underline{7}\ \underline{E}\ \underline{E}\ \underline{7}\ \underline{7}\ \underline{7}$ $\quad |X| = \binom{9}{3} 7^6$

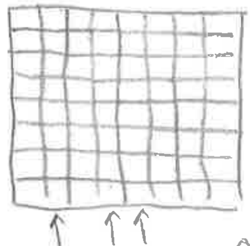
Y: $\underline{E}\ \underline{7}\ \underline{7}\ \underline{7}\ \underline{E}\ \underline{E}\ \underline{7}\ \underline{7}\ \underline{7}\ \underline{7}$ $\quad |Y| = \binom{9}{2} 7^7$

By addition principle, answer is

$$|X| + |Y| = \boxed{\binom{9}{3} 7^6 + \binom{9}{2} 7^7}$$

2. In how many ways can three black rooks and three white rooks be placed on an 8×8 chessboard so that none can attack any other? (For example, see test cover page.)

First choose $\binom{8}{3}$ columns for the black rook.
Then choose $\binom{5}{3}$ of the remaining columns for the white rook,



Now consider these chosen columns from left to right. There are 8 rows for the left-most rook, 7 rows for the next, 6 rows for the one after that, etc. Using the multiplication principle our answer is

$$\boxed{\binom{8}{3}\binom{5}{3} 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 22{,}579{,}200}$$

3. A bag contains 10 identical red marbles, 10 identical blue marbles, 10 identical green marbles and 10 identical white marbles. You take out 10 marbles. How many possible outcomes are there?

Look at the bag of marbles as a multiset $S = \{ \underset{10}{R}, \underset{10}{B}, \underset{10}{G}, \underset{10}{W} \}$

The question is asking how many 10-combinations does S have.

$$\underbrace{** \cdots *}_{R} \,|\, \underbrace{** \cdots *}_{B} \,|\, \underbrace{** \cdots *}_{G} \,|\, \underbrace{** \cdots *}_{W}$$

By the stars and bars technique, the answer is

$$\boxed{\binom{10+3}{10} = \binom{13}{10} = \frac{13 \cdot 12 \cdot 11}{3!} = 286}$$

4. How many anagrams of the word INDEPENDENT are there?

11 letters, 2 D's
3 N's and 3 E's

Answer: $\binom{11}{3\ 3\ 2\ 1\ 1\ 1} = \dfrac{11!}{3!\,3!\,2!}$

$= \boxed{554400}$

5. This question concerns the expansion of $(w - x + 2y + z)^{20}$.

(a) What is the coefficient of the term $w^4x^7y^3z^6$?

By multinomial theorem, term is

$\binom{20}{4\ 7\ 3\ 6} w^4(-x)^7(2y)^3 z^6$

$= \dfrac{20!}{4!\,7!\,3!\,6!}(-1)^7 2^3 w^4 x^7 y^3 z^6$

Answer: $\boxed{\dfrac{-8 \cdot 20!}{4!\,7!\,3!\,6!}}$

(b) How many terms does the expansion have?

Each such term has form
$w^a x^b y^c z^d$ where $a+b+c+d=20$
and each is non-negative.

$\underbrace{***}_{a} \mid \underbrace{***}_{b} \mid \underbrace{****}_{c} \mid \underbrace{***}_{d}$

By stars and bars, The number of such a, b, c, d values is

$\binom{20+3}{20} = \boxed{\binom{23}{20}}$

6. Prove that for any $n+1$ distinct integers $a_1, a_2, a_3, \ldots, a_{n+1}$, there are two of them whose difference is a multiple of n.

Let $X = \{a_1, a_2, a_3, \ldots, a_{n+1}\}$ and $Y = \{0, 1, 2, \ldots, n-1\}$. Note that $|X| = n+1$ and $|Y| = n$, so $|X| > |Y|$. Now let $f : X \rightarrow Y$ be the function $f(a_i) = a_i \pmod{n}$, i.e. $f(a_i) = \left(\begin{array}{c}\text{remainder when} \\ \text{dividing } a_i \text{ by } n\end{array}\right)$

By the pigeonhole principle, f is <u>not</u> injective. Thus there exist distinct a_i & a_j with $f(a_i) = f(a_j)$; that is $a_i \equiv a_j \pmod{n}$

This means $a_i - a_j$ is a multiple of n.

7. Given a standard 52-card deck, how many 4-card hands are there with all 4 cards of different suits or all 4 face cards (J,K,Q)?

Let A = set of 4-card hands with all 4 cards of different suits. By the mult. principle $|A| = 13^4$

♡ ◇ ♣ ♠
$\boxed{13}\ \boxed{13}\ \boxed{13}\ \boxed{13}$

Let B be the set of 4-card hands with all 4 cards face cards, J, K, Q. Then $|B| = \binom{12}{4}$ because there are 12 face cards.

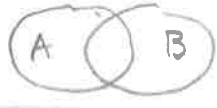
Then $A \cap B$ is the set of 4-card hands consisting of face cards of different suits.

Then $|A \cap B| = 3^4$

♡ ◇ ♣ ♠
$\boxed{3}\ \boxed{3}\ \boxed{3}\ \boxed{3}$

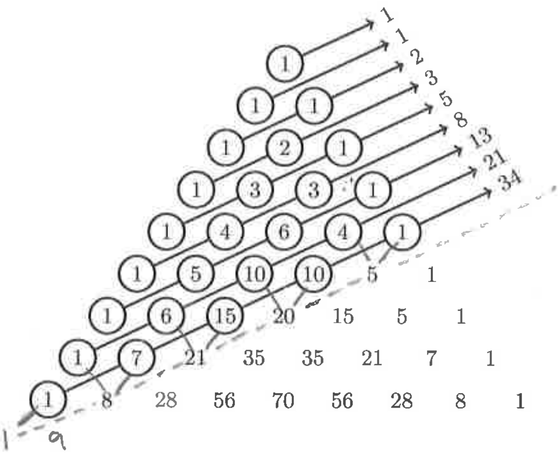
By inclusion - exclusion our answer is

$|A \cup B| = |A| + |B| - |A \cap B|$



$$= 13^4 + \binom{12}{4} - 3^4 = \boxed{28975}$$

8. The indicated diagonals of Pascal's triangle sum to Fibonacci numbers. Explain why this pattern continues forever.



This follows from Pascal's formula $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$. Notice that, as indicated above, any entry of a diagonal is the sum of the two entries a and b immediately above it:

Note that a and b belong to the two previous diagonals, respectively. Consequently the sum of the entries in a diagonal is the sum of the entries in the two previous diagonals. Hence it is a Fibonacci number.

Fibonacci numbers are those in the sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$ where any term is the sum of the prior two terms.

9. Prove that $\binom{n}{1} - 2\binom{n}{2} + 3\binom{n}{3} - 4\binom{n}{4} + \cdots + (-1)^{n-1}n\binom{n}{n} = 0.$

Suggestion: Start with the binomial theorem.

Begin with the binomial theorem in the form

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + \binom{n}{n}x^n$$

Differentiating both sides produces

$$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + 4\binom{n}{4}x^3 + \cdots + n\binom{n}{n}x^{n-1}$$

Next, plug in $x=-1$ to get

$$n(1-1)^{n-1} = \binom{n}{1} - 2\binom{n}{2} + 3\binom{n}{3} - 4\binom{n}{4} + \cdots + n\binom{n}{n}(-1)^{n-1}$$

or

$$0 = \binom{n}{1} - 2\binom{n}{2} + 3\binom{n}{3} - 4\binom{n}{4} + \cdots + (-1)^{n-1}n\binom{n}{n}.$$

QED

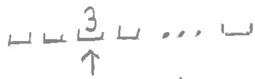
10. Prove: $\binom{n}{0}D_n + \binom{n}{1}D_{n-1} + \binom{n}{2}D_{n-2} + \binom{n}{3}D_{n-3} + \cdots + \binom{n}{n}D_0 = n!$

(Here D_n is the nth derangement number.)

<u>Proof</u> Let $S = \{1, 2, 3, \ldots, n\}$. Observe that:

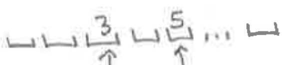
$$\binom{n}{0}D_n = D_n = \left(\begin{array}{l}\text{number of permutations of } S \text{ in which} \\ \text{no number is in its natural position}\end{array}\right)$$

$$\binom{n}{1}D_{n-1} = nD_{n-1} = \left(\begin{array}{l}\text{number of permutations of } S \text{ in which} \\ \text{exactly 1 number is in its natural pos.}\end{array}\right)$$



<u>Reason:</u> There are n ways to choose a number to be in its natural position, then D_{n-1} ways to re-arrange the other $n-1$, with no number in its natural position

$$\binom{n}{2}D_{n-2} = \left(\begin{array}{l}\text{\# of permutations of } S \text{ with} \\ \text{exactly 2 numbers in their natural position}\end{array}\right)$$



Reason: There are $\binom{n}{2}$ ways to choose 2 numbers to be in their natural position, and D_{n-2} ways to re-arrange the remaining $n-2$ numbers

$\vdots$

$$\binom{n}{k}D_{n-k} = \left(\begin{array}{l}\text{\# of permutations of } S \text{ with} \\ \text{exactly } k \text{ \#'s in their natural pos.}\end{array}\right)$$

$\vdots$

$$\binom{n}{n}D_0 = 1 = \left(\begin{array}{l}\text{\# of permutations of } S \text{ with} \\ \text{exactly } n \text{ \#'s in natural positions}\end{array}\right)$$

Now, the total number of permutations of S is $n!$. By the addition principle we can also count them by adding up the above terms. That is,

$$n! = \binom{n}{0}D_n + \binom{n}{1}D_{n-1} + \binom{n}{2}D_{n-2} + \cdots + \binom{n}{n}D_0.$$