VCU

MATH 525
COMBINATORICS

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TEST 1

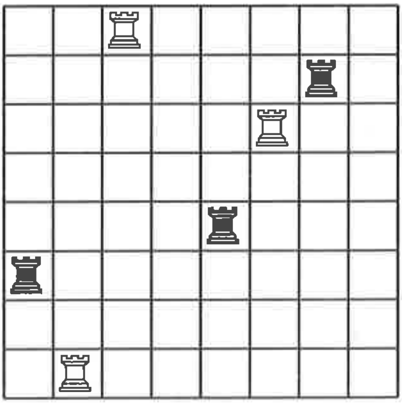
February 25, 2016

Name: ______________________________

Directions. Answer the questions in the space provided. Justify each step to the extent reasonable.

This is a closed-book, closed-notes test.

There are 10 numbered questions; each is worth 10 points.



1. Consider 10-letter strings made from the letters A, B, C, D, E, F, G, H.

   (a) How many strings are there (repetition allowed) for which no two consecutive letters are the same?

   (b) How many strings are there (repetition allowed) for which the first entry is a vowel, and the string contains exactly three E's?

2. In how many ways can three black rooks and three white rooks be placed on an 8×8 chessboard so that none can attack any other? (For example, see test cover page.)

3. A bag contains 10 identical red marbles, 10 identical blue marbles, 10 identical green marbles and 10 identical white marbles. You take out 10 marbles. How many possible outcomes are there?

4. How many anagrams of the word INDEPENDENT are there?

5. This question concerns the expansion of $(w - x + 2y + z)^{20}$.

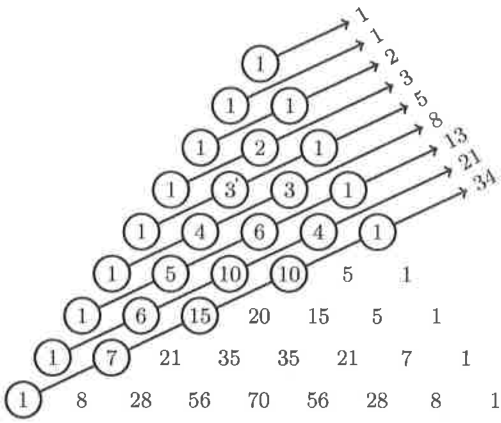
(a) What is the coefficient of the term $w^4x^7y^3z^6$?

(b) How many terms does the expansion have?

6. Prove that for any $n + 1$ distinct integers $a_1, a_2, a_3, \ldots, a_{n+1}$, there are two of them whose difference is a multiple of n.

7. Given a standard 52-card deck, how many 4-card hands are there with all 4 cards of different suits or all 4 face cards (J,K,Q)?

8. The indicated diagonals of Pascal's triangle sum to Fibonacci numbers. Explain why this pattern continues forever.



Fibonacci numbers are those in the sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$ where any term is the sum of the prior two terms.

9. Prove that $\binom{n}{1} - 2\binom{n}{2} + 3\binom{n}{3} - 4\binom{n}{4} + \cdots + (-1)^{n-1}n\binom{n}{n} = 0.$

Suggestion: Start with the binomial theorem.

10. Prove: $\binom{n}{0}D_n + \binom{n}{1}D_{n-1} + \binom{n}{2}D_{n-2} + \binom{n}{3}D_{n-3} + \cdots + \binom{n}{n}D_0 = n!$

(Here D_n is the nth derangement number.)