Name: KEY

MATH 525 – FINAL EXAM
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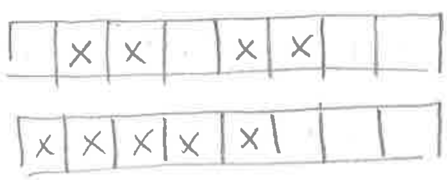
Directions. Answer the questions in the space provided. This is a closed-notes, closed book exam; no calculators, no computers and no formula sheets. You have three hours.

(1) A classroom has 2 rows of 8 seats each. There are 14 students, 5 of whom always sit in the first row and 4 of whom always sit in the back row. In how many ways can the students be seated?



$\leftarrow \binom{8}{4}$ choices of seats for back-row students

$\leftarrow \binom{8}{5}$ choices of seats for front-row students.

You can fill the classroom in the following ways.

$$\binom{8}{4} 4! \binom{8}{5} 5! \binom{7}{5} 5!$$

- $\binom{8}{4}$: Choose 4 seats in back for back row students
- $4!$: ways to arrange back row students in 4 chosen seats
- $\binom{8}{5}$: choose 5 seats in front for the front-row students
- $5!$: ways to arrange front row students in 5 chosen seats
- $\binom{7}{5}$: Now there are 5 students left and 7 seats to put them in. Choose $\binom{7}{5}$ seats for them
- $5!$: ways to arrange the 5 remaining students in 5 chosen chairs

$$= \frac{8!}{4!\cancel{4!}}\cancel{4!}\,\frac{8!}{\cancel{5!}3!}5!\,\frac{7!}{2!\cancel{5!}}\cancel{5!} = \frac{8!\ 8!\ 7!}{4!\ 3!\ 2!}$$

$$= (8\cdot7\cdot6\cdot5)\cdot(8\cdot7\cdot6\cdot5\cdot4)\cdot(7\cdot6\cdot5\cdot4\cdot3)$$

$$= 28{,}449{,}792{,}000$$

(2) (a) Find the coefficient of $x^3y^2z^5$ in $(x+y+z)^{10}$.

$$\frac{10!}{3!\,2!\,5!} = \frac{10\cdot 9\cdot \overset{4}{\cancel{8}}\cdot 7\cdot \cancel{6}}{\cancel{2}\quad \cancel{3!}}$$

$$= 10\cdot 9\cdot 4\cdot 7 = \boxed{2520}$$

(By multinomial theorem)

(b) Find the number of integer solutions of $x+y+z+w=30$ that satisfy $x \geqslant 2$, $y \geqslant 0$, $z \geqslant -8$, $w \geqslant 5$.

This is the same as:

$$x-2+y+z+8+w-5 = 30-2+8-5$$

$$(x-2)+y+(z+8)+(w-5) = 31$$

$$a+b+c+d = 31$$

let $a = x-2$
$b = y$
$c = z+8$
$d = w+5$

Now we need non-neg integer solutions of the above.

$$\overbrace{****}^{a}\ |\ \overbrace{**\cdots*}^{b}\ |\ \overbrace{**\cdots*}^{c}\ |\ \overbrace{***\cdots*}^{d}$$

This is the number of ways to choose 3 spots out of $31+3=34$ spots for bars.

Answer: $\binom{34}{3} = \binom{34}{31} = \dfrac{34\cdot 33\cdot 32}{3!} = 34\cdot 11\cdot 16$

$= \boxed{5984}$

(3) There are 20 sticks lined up in a row occupying 20 distinct positions:

||||||||||||||||||||,

and six of them are to be chosen. How many choices are there if there must be at least two sticks between any pair of chosen sticks?

a | b | c | d | e | f | g

Let $a, b, c, \dots g$ be the number of sticks in the gaps between the six chosen sticks, as show above. The problem requires

$a \geq 0, \quad b \geq 2, \quad c \geq 2, \quad d \geq 2, \quad e \geq 2, \quad f \geq 2, \quad g \geq 0.$

Since there are 14 sticks total in the gaps we must have

$$a + b + c + d + e + f + g = 14$$

$$a + (b-2) + (c-2) + (d-2) + (e-2) + (f-2) + g = 14 - 10$$

Let $b' = b-2, \quad c' = c-2, \quad d' = d-2, \quad e' = e-2 \quad f' = f-2$

So the above becomes

$$a + b' + c' + d' + e' + f' + g = 4$$

and we seek the number of non-negative integer solutions, which is

$$\binom{4+6}{6} = \binom{10}{6} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2} = 10 \cdot 3 \cdot 7 = \boxed{210}$$

(4) Use a combinatorial argument to prove that $\sum_{k=0}^{n} \binom{a}{k}\binom{b}{n-k} = \binom{a+b}{n}$ for all positive integers a, b and n.

Suppose we have a red balls and b blue balls. Then the total number of ways to select n of these balls is $\binom{a+b}{n}$. This is the right hand side.

Now let's count it a different way. To select n of the balls, we take k of the red ones and n-k of the blue ones, where $0 \leq k \leq n$. By the multiplication principle the number of ways to do this is

$$\binom{a}{k}\binom{b}{n-k}.$$

Notice that even when $k > a$ (i.e. it exceeds the number of red balls) this still makes sence because then $\binom{a}{k} = 0$, so $\binom{a}{k}\binom{b}{n-k} = 0$, meaning there are 0 ways to select more red balls than exist.

As k (the number of red balls) might be anywhere between 0 and n, the total number of ways to select n is

$$\binom{a}{0}\binom{b}{n} + \binom{a}{1}\binom{b}{n-1} + \binom{a}{2}\binom{b}{n-2} + \cdots \binom{a}{n}\binom{b}{0}$$

$$= \sum_{k=0}^{n} \binom{a}{k}\binom{b}{n-k}.$$ Therefore $\sum_{k=0}^{n} \binom{a}{k}\binom{b}{n-k} = \binom{a+b}{n}$

because both sides count the same thing.

(5) Find and verify a formula for $\sum_{\substack{k,\ell,m \geqslant 0 \\ k+\ell+m=n}} \binom{a}{k}\binom{b}{\ell}\binom{c}{m}$.

[Suggestion: generalize your argument from question 4 on the previous page.]

We claim $\sum_{\substack{k,\ell,m \geq 0 \\ k+\ell+m=n}} \binom{a}{k}\binom{b}{\ell}\binom{c}{m} = \binom{a+b+c}{n}$.

Proof Imagine we have a red balls, b blue balls and c white balls. Thus we have a+b+c balls all together. The number of ways to select n of these balls is therefore the right hand side

$$\binom{a+b+c}{n}.$$

Now let's count this a different way. In selecting n balls we take k red balls, ℓ blue balls and m white balls, for some $k, \ell, m \geq 0$ and $k+\ell+m=n$. By the multiplication principle there are $\binom{a}{k}\binom{b}{\ell}\binom{c}{m}$ ways to do this. Summing over all possibilities for k, ℓ, m, we see that the total number of ways to select n of the balls is

$$\sum_{\substack{k,\ell,m \geq 0 \\ k+\ell+m=n}} \binom{a}{k}\binom{b}{\ell}\binom{c}{m}.$$

Therefore we must conclude $\sum_{\substack{k,\ell,m \geq 0 \\ k+\ell+m=n}} \binom{a}{k}\binom{b}{\ell}\binom{c}{m} = \binom{a+b+c}{n}$ because both sides count the same thing.

(6) Find the number of integers between 1 and 10,000 (inclusive) which are *not* divisible by 4, 5, or 6.

Let A be set of numbers between 1 & 10000 that are multiples of 4
Let B " " " " " " " " " multiples of 5
Let C " " " " " " " " " " 6

Thus $|A| = \frac{10000}{4} = 2500$

$|B| = \frac{10000}{5} = 2000$

$|C| = \left\lfloor \frac{10000}{6} \right\rfloor = 1666$

Also $|A \cap B| = \frac{10000}{\operatorname{lcm}(4,5)} = \frac{10000}{20} = 500$

$|A \cap C| = \left\lfloor \frac{10000}{\operatorname{lcm}(4,6)} \right\rfloor = \left\lfloor \frac{10000}{12} \right\rfloor = 833$

$|B \cap C| = \left\lfloor \frac{10000}{\operatorname{lcm}(5,6)} \right\rfloor = \left\lfloor \frac{10000}{30} \right\rfloor = 333$

And $|A \cap B \cap C| = \left\lfloor \frac{10000}{\operatorname{lcm}(4,5,6)} \right\rfloor = \left\lfloor \frac{10000}{60} \right\rfloor = 166$

By inclusion - exclusion, the answer to the question is

$10000 - |A \cup B \cup C|$

$= 10000 - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C|$

$= 10000 - 2500 - 2000 - 1666 + 500 + 833 + 333 - 166$

$= \boxed{5334}$

(7) Consider the multiset $X = \{\infty \cdot a,\ \infty \cdot b,\ \infty \cdot c,\ \infty \cdot d\}$. Let h_n be the number of n-combinations of X that have no more than 4 a's, a multiple of 5 b's, at least 5 c's, and at least 2 d's.

(a) Find an ordinary generating function for h_n.

(b) Use your answer from part (a) to find a general formula for h_n.

$$g(x) = \underbrace{(1+x+x^2+x^3+x^4)}_{\text{choices for } \underline{a}}\underbrace{(1+x^5+x^{10}+x^{15}+\cdots)}_{\text{choices for } \underline{b}}\underbrace{(x^5+x^6+x^7+x^8+\cdots)}_{\text{choices for } \underline{c}}\underbrace{(x^2+x^3+x^4+x^5)}_{\text{choices for } \underline{d}}$$

$$= \frac{1-x^5}{1-x}\,\frac{1}{1-x^5}\,x^5(1+x+x^2+\cdots\,]\,x^2(1+x+x^2+\cdots)$$

$$= \frac{1-x^5}{1-x}\,\frac{1}{1-x^5}\,x^7\,\frac{1}{1-x}\,\frac{1}{1-x} = \boxed{\frac{x^7}{(1-x)^3}}$$

$$g(x) = x^7\,\frac{1}{(1-x)^3} = x^7 \sum_{k=0}^{\infty} \binom{3+k-1}{k} x^k$$

$$= \sum_{k=0}^{\infty} \binom{k+2}{k} x^{k+7}$$

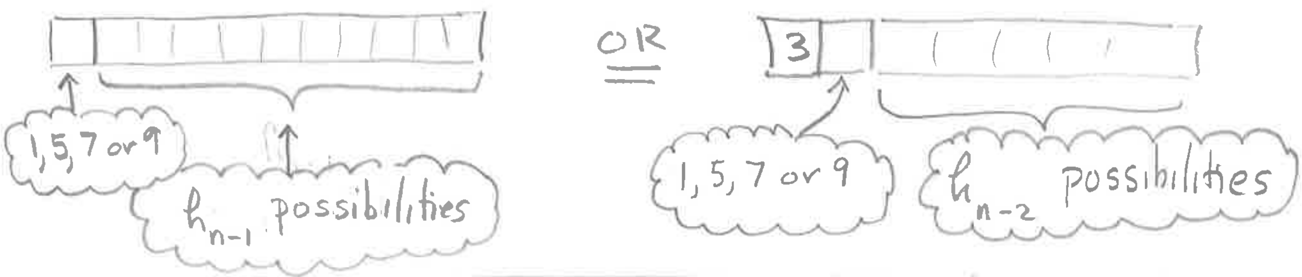
$$= \sum_{n=0}^{\infty} \binom{n-5}{n-7} x^n$$

Therefore $\boxed{h_n = \binom{n-5}{n-7}}$

(8) Let h_n be the number of n-digit numbers with all digits odd and no consecutive 3's. (By default, assume $h_0 = 1$.)

(a) Find a homogeneous recurrence relation for h_n.
(b) Solve the recurrence relation to find a general formula for h_n.

Such n-digit numbers look like:



Therefore $\boxed{h_n = 4h_{n-1} + 4h_{n-2} \quad \text{with } h_0 = 1 \text{ and } h_1 = 5}$

$$h_n - 4h_{n-1} - 4h_{n-2} = 0$$

$$x^2 - 4x - 4 = 0 \leftarrow \text{(characteristic equation)}$$

$$x = \frac{4 \pm \sqrt{4^2 + 4 \cdot 4}}{2} = \frac{4 \pm \sqrt{32}}{2} = \frac{4 \pm 4\sqrt{2}}{2} = 2 \pm 2\sqrt{2} \leftarrow \text{(roots)}$$

Therefore $h_n = a(2+2\sqrt{2})^n + b(2-2\sqrt{2})^n$

$\underline{n=0:}$ $\quad 1 = h_0 = a(2+2\sqrt{2})^0 + b(2-2\sqrt{2})^0 \Rightarrow 1 = a+b$

$\underline{n=1:}$ $\quad 5 = h_1 = a(2+2\sqrt{2})^1 + b(2-2\sqrt{2})^1$

$$5 = 2a + 2\sqrt{2}a + 2b - 2\sqrt{2}b$$
$$5 = 2(a+b) + 2\sqrt{2}(a-b)$$
$$5 = 2 + 2\sqrt{2}(1-2b)$$
$$5 = 2 + 2\sqrt{2} - 4\sqrt{2}b$$
$$3 - 2\sqrt{2} = -4\sqrt{2}b \Rightarrow \boxed{b = \frac{3-2\sqrt{2}}{-4\sqrt{2}} = \frac{4-3\sqrt{2}}{8}}$$

$$\boxed{a = \frac{4+3\sqrt{2}}{8}}$$

$\underline{\text{Answer:}}$ $\boxed{h_n = \frac{4+3\sqrt{2}}{8}(2+2\sqrt{2})^n + \frac{4-3\sqrt{2}}{8}(2-2\sqrt{2})^n}$

(9) Let h_n be the number of n-digit numbers with all digits odd and for which the digits 1 and 3 occur a *positive* even number of times.

(a) Find an exponential generating function for h_n.

(b) Use your answer from part (a) to find a general formula for h_n.

$$g^{(e)}(x) = \underbrace{\left(\frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\right)^2}_{\text{choices for } 1,\, 3} \underbrace{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^3}_{\text{choices for } 5,\, 7,\, 9}$$

$$= \left(\frac{e^x + e^{-x}}{2} - 1\right)^2 (e^x)^3$$

$$= \left(\frac{1}{4}(e^{2x} + 2 + e^{-2x}) - 2\,\frac{e^x + e^{-x}}{2} + 1\right)(e^x)^3$$

$$= \left(\frac{1}{4}e^{2x} + \frac{1}{2} + \frac{1}{4}e^{-2x} - e^x - e^{-x} + 1\right) e^{3x}$$

$$= \left(\frac{1}{4}e^{2x} - e^x + \frac{3}{2} - e^{-x} + \frac{1}{4}e^{-2x}\right) e^{3x}$$

$$= \frac{1}{4}e^{5x} - e^{4x} + \frac{3}{2}e^{3x} - e^{2x} + \frac{1}{4}e^x$$

$$= \sum_{n=1}^{\infty}\left(\frac{1}{4}5^n - 4^n + \frac{3}{2}3^n - 2^n + \frac{1}{4}\right)\frac{x^n}{n!}$$

Thus $\boxed{h_n = \frac{1}{4}5^n - 4^n + \frac{1}{2}3^{n+1} - 2^n + \frac{1}{4}}$

(10) The general term of a sequence $h_0, h_1, h_2, \ldots$ is a polynomial in n of degree 3. The first four entries of the 0th diagonal of its difference table are $1, -1, 3, 10$. Find the formula for h_n.

$$h_n = 1\binom{n}{0} - 1\binom{n}{1} + 3\binom{n}{2} + 10\binom{n}{3}$$

$$= 1 - n + 3\frac{n(n-1)}{2} + 10\frac{n(n-1)(n-2)}{3!}$$

$$= 1 - n + \frac{3}{2}n^2 - \frac{3}{2}n + \frac{5}{3}(n^3 - 3n^2 + 2n)$$

$$\boxed{h_n = 1 + \frac{5}{6}n - \frac{7}{2}n^2 + \frac{5}{3}n^3}$$

Gives:

1		0		2		17		55		126	
	-1		2		15		38		71		...
		3		13		23		33		...	
			10		10		10		...		
				0		0		...			
					0		...				